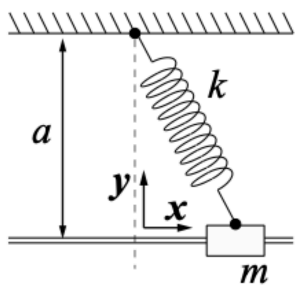
Physics Placement Exam: Classical Mechanics and Electromagnetism

07-Sep-23

Please do all six problems. Generous partial credit may be awarded to partial solutions provided your work is organized and legible. Note that various formulas that may be useful are at the end of the exam.

You must not use your phone, calculator, or any other device with messaging capabilities during the exam.

Problem 1: A mass m is free to slide along a frictionless horizontal rail. A spring with spring constant k and rest length $2a$ is attached to the mass such that one end moves with the mass, and the other is fixed to the ceiling above. The ceiling is located at perpendicular height a from the rail such that when the spring is oriented vertically, it is compressed to half its rest length. Ignore any gravity effects in this problem.

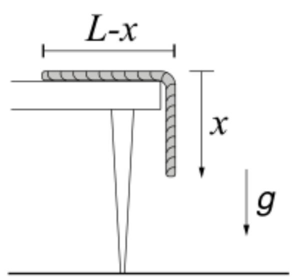


a) Write down an expression for the potential energy of the system as a function of the displacement of the mass along the rail. Take the elastic potential to be zero valued when the spring is unstretched.

b) Find any equilibrium positions and identify whether they are stable or unstable. *[Hint: there are two stable and one unstable points.]*

c) Imagine the mass is at rest at one of the stable points, when it is given a speed, instantaneously. What minimum speed would be required in order for the mass to travel all the way to the other stable point?

Problem 2: A frictionless, uniform rope of total length L and mass m is placed on a table such that initially a portion of the rope, length l, hangs vertically off the edge. Imagine that the rope is released from rest and assume the hanging portion always remains vertical.



a) Using the length of the rope that hangs vertically as the generalized coordinate (see Fig.), write down a Lagrangian for this system.

b) Find an expression for the acceleration of the rope.

c) Find an expression for the length of rope that hangs off the table versus time.

Problem 3: The Lagrangian for a physical system is given by

$$L = \frac{m}{2}\dot{x}^2 e^{\lambda t} \ . \tag{1}$$

Using only conserved quantities, find a general solution to the motion of the system described by the Lagrangian.

Problem 4: A spherical ball of radius a contains a charge density on the surface given by $\sigma(\theta) = \sigma_0(1 - \cos\theta)$ where θ is the conventional polar angle in spherical coordinates and σ_0 is a positive constant.

For parts (a) -(c), assume that the shell is non-conducting, and no other charges exist in space.
(a) Calculate the potential for $r > a$.
(b) Calculate the electric field everywhere in space and draw the field lines.
(c) Calculate all the multipole moments of the charge distribution (hint: use your result from part (a) to make the calculation quick) .

For parts (d)-(e) assume that the sphere is conducting. Assume that the charge density given above is on the outer surface of the sphere.
(d) Calculate the electric potential for $r > a$.
(e) What is the electric field everywhere in space? What part of this field is applied external to the conducting shell?

Problem 5: A thin disk of radius R has its normal along z. This disk has a uniform surface charge σ and rotates at a constant angular velocity ω in the counterclockwise direction when viewed from a point on the positive z-axis.
(a) Write down an expression for the surface current density $\mathbf{K}$ of the disk as a function of cylindrical coordinates. Make sure to specify both the magnitude and direction.
(b) Find an expression for the magnetic field on the z-axis for $z > 0$.
(c) Show that the results agree with the form of a magnetic dipole at long distances. Identify the magnitude of the magnetic dipole moment. You may make use of the expansion for $\epsilon \ll 1$,

$$\frac{1}{\sqrt{1+\epsilon}} = 1 - \frac{1}{2}\epsilon + \frac{3}{8}\epsilon^2 + \dots$$

(d) Calculate the magnetic dipole moment of the disk directly and show that it agrees with your answer in part (c) .

Problem 6: An electric dipole at the origin is oscillating in magnitude with an angular frequency ω so that the dipole moment is given by $\mathbf{p} = p_0 \cos\omega t\, \hat{\mathbf{z}}$.
(a) Explain how conservation of energy requires, at large distances, that both the electric and magnetic field must decrease with distance from the origin as $1/r$. In particular, explain why the fields cannot decrease as $1/r^\alpha$ with $\alpha \neq 1$.
(b) At large distance from the origin, the magnetic field, in spherical coordinates, is given by

$$\mathbf{B}(\mathbf{r}, t) = -\frac{\mu_0 p_0 \omega^2}{4\pi c} \sin\theta \frac{\cos\omega(t - r/c)}{r}\ \hat{\phi}$$

Write down the full expression for the electric field $\mathbf{E}(\mathbf{r}, t)$ at large distances. Briefly justify your answer.
(c) How does the power emitted by the dipole vary as a function of the angle θ between $\mathbf{p}$ and the line of sight of an observer? Illustrate the power pattern $d\langle P\rangle/d\Omega$ with a simple drawing, where $\langle P\rangle$ is the time-averaged power radiated by the dipole.

Potentially Useful Equations and Definitions

Noether theorem: In classical mechanics, if a Lagrangian $L(x, \dot{x})$ is invariant under an infinitesimal transformation

$$t \to t + \epsilon \Delta t \ , \qquad x \to x + \epsilon \Delta x \tag{2}$$

for given Δt and Δx and for arbitrary (infinitesimal ϵ), then the quantity

$$Q = \frac{\partial L}{\partial \dot{x}} \Delta x + \left(L - \frac{\partial L}{\partial \dot{x}} \dot{x} \right) \Delta t \tag{3}$$

is conserved.

Cylindrical coordinates: $x = s\cos\phi$, $y = s\sin\phi$, $z = z$, $\quad s = \sqrt{x^2 + y^2}$

$$\begin{aligned}
\nabla t &= \frac{\partial t}{\partial s}\,\hat{\mathbf{s}} + \frac{1}{s}\frac{\partial t}{\partial \phi}\,\hat{\phi} + \frac{\partial t}{\partial z}\,\hat{\mathbf{z}} \\
\nabla \cdot \mathbf{v} &= \frac{1}{s}\frac{\partial (s v_s)}{\partial s} + \frac{1}{s}\frac{\partial v_\phi}{\partial \phi} + \frac{\partial v_z}{\partial z} \\
\nabla \times \mathbf{v} &= \left[\frac{1}{s}\frac{\partial v_z}{\partial \phi} - \frac{\partial v_\phi}{\partial z}\right]\hat{\mathbf{s}} + \left[\frac{\partial v_s}{\partial z} - \frac{\partial v_z}{\partial s}\right]\hat{\phi} + \frac{1}{s}\left[\frac{\partial (s v_\phi)}{\partial s} - \frac{\partial v_s}{\partial \phi}\right]\hat{\mathbf{z}} \\
\nabla^2 t &= \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial t}{\partial s}\right) + \frac{1}{s^2}\frac{\partial^2 t}{\partial \phi^2} + \frac{\partial^2 t}{\partial z^2}
\end{aligned}$$

Spherical coordinates: $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$.

$$\begin{aligned}
\nabla t &= \frac{\partial t}{\partial r}\,\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial t}{\partial \theta}\,\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial t}{\partial \phi}\,\hat{\phi} \\
\nabla \cdot \mathbf{v} &= \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta\, v_\theta) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi} \\
\nabla \times \mathbf{v} &= \frac{1}{r\sin\theta}\left[\frac{\partial(\sin\theta\, v_\phi)}{\partial \theta} - \frac{\partial v_\theta}{\partial \phi}\right]\hat{\mathbf{r}} + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial v_r}{\partial \phi} - \frac{\partial (r v_\phi)}{\partial r}\right]\hat{\theta} + \frac{1}{r}\left[\frac{\partial (r v_\theta)}{\partial r} - \frac{\partial v_r}{\partial \theta}\right]\hat{\phi} \\
\nabla^2 t &= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial t}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial t}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 t}{\partial \phi^2}
\end{aligned}$$

Solutions when there is no ϕ dependence:

$$\Phi(r,\theta) = \sum_{l=0}^{\infty}\left[A_l\, r^l + \frac{B_l}{r^{l+1}}\right] P_l(\cos\theta) \quad \text{satisfies} \quad \nabla^2\Phi = 0 \quad .$$

$$P_0(u) = 1 \ , \ P_1(u) = u \ , \ P_2(u) = \frac{3}{2}u^2 - \frac{1}{2} \ , \ P_3(u) = \frac{5}{2}u^3 - \frac{3}{2}u$$

$$\int_{-1}^{1} P_m(u) P_n(u)\, du = \frac{2}{2n+1}\,\delta_{m,n}$$

Electrostatic energy (W):

Discrete charges:

$$W = \frac{1}{8\pi\epsilon_0} \sum_{i=1}^{n} \sum_{j \neq i}^{n} \frac{q_i q_j}{r_{ij}}$$

Continuous charge distribution:

$$W = \frac{\epsilon_0}{2} \int \mathbf{E}(\mathbf{r}) \cdot \mathbf{E}(\mathbf{r})\, d^3r = \frac{1}{2} \int \rho(\mathbf{r}) \Phi(\mathbf{r})\, d^3r$$

Field from magnetic dipole:

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} \frac{\mathbf{3m} \cdot \hat{\mathbf{r}} - \mathbf{m}}{r^3}$$

Energy density and flux, momentum density:

$$u = \frac{1}{2}\epsilon_0\, \mathbf{E} \cdot \mathbf{E} + \frac{1}{2\mu_0} \mathbf{B} \cdot \mathbf{B} \quad , \quad \mathbf{S} = \frac{1}{\mu_0} \mathbf{E} \times \mathbf{B} \quad , \quad \mathbf{g} = \epsilon_0 (\mathbf{E} \times \mathbf{B})$$

Speed of light, impedance of the vacuum:

$$\mu_0 \epsilon_0 = \frac{1}{c^2} \quad , \quad \mu_0 c = 377\ \Omega$$

Physics Placement Exam
Quantum Mechanics, Statistical Mechanics and Thermodynamics
08-Sep-23

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Problem 1:

A spin-1 particle with charge q is in a uniform magnetic field of magnitude B directed along the +z axis. At time $t = 0$ the particle is known to be in a state satisfying $\hat{S}_x|\psi\rangle = \hbar|\psi\rangle$.

(a) Find $|\psi(t)\rangle$.

(b) Find the time dependence of the expectation value for all three components of the spin operator $\hat{\mathbf{S}}$.

Problem 2:

Consider the Hamiltonian:

$$\hat{H} = \frac{\hat{p}_x^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2 + \lambda\hat{p}_x^2.$$

We can rewrite this Hamiltonian as $\hat{H} = \hat{H}_0 + \hat{V}$, such that

$$\hat{H}_0 = \frac{\hat{p}_x^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2$$
$$\hat{V} = \lambda\hat{p}_x^2,$$

with λ being very small. For this problem, work in the eigenstates such that $\hat{H}_0\,|n\rangle = E_n\,|n\rangle$.

(a) What is the first order correction to the ground state energy?

(b) What is the first order correction to the ground state wavefunction?

(c) Calculate the energy spectrum exactly and compare your results in (a).

Problem 3:

Consider the following operators in a complete orthonormal basis of pure states $|1\rangle$ and $|2\rangle$ of the Hilbert space:

$$\rho_A = \begin{pmatrix} 1 & i \\ -i & 0 \end{pmatrix}, \quad \rho_B = \begin{pmatrix} 1/2 & i/2 \\ i/2 & i/2 \end{pmatrix}, \quad \rho_C = \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix}, \quad \rho_D = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}.$$

(a) Which operators are admissible density matrices?

(b) Which ones describe mixed states?

(c) Which ones describe pure states? In case there is any pure state, write it as a linear superposition of $|1\rangle$ and $|2\rangle$.

Problem 4:

(a. 5 points) Let x and y be random variables drawn from the two independent Gaussian distributions

$$\begin{aligned} P_1(x) &= \frac{1}{\sqrt{2\pi}\sigma_x} \exp\left[-\frac{(x-\mu_x)^2}{2\sigma_x^2}\right], \\ P_2(y) &= \frac{1}{\sqrt{2\pi}\sigma_y} \exp\left[-\frac{(y-\mu_y)^2}{2\sigma_y^2}\right]. \end{aligned}$$

Let $z = x + y$. What is its mean $\langle z \rangle$ and its variance $\sqrt{\langle (z - \langle z \rangle)^2 \rangle}$? Write down the expression for the probability distribution function $P(z)$. Can you give a proof for the last step?

(b. 5 points). Consider a mixture of two monoatomic ideal gases a and b at temperature T, with m_a and m_b being the masses of their atoms, respectively. The Mawell-Boltzmann distribution of the gas particles' velocity vectors $\vec{v} = (v_x, v_y, v_z)$ is given by

$$P(v_x, v_y, v_z) = \frac{1}{(2\pi)^{3/2} v_{\rm th}^3} \exp\left[-\frac{\vec{v}^2}{2v_{\rm th}^2}\right], \tag{0.0.1}$$

where

$$v_{\rm th} = \sqrt{k_B T/m}. \tag{0.0.2}$$

Here k_B is the Boltzmann constant and m is the mass of the atom. Consider now a randomly drawn pair of atoms, one from from a and the other one from b. Let their relative velocity be $\vec{v}_{\rm rel} = \vec{v}_a - \vec{v}_b$. Find the probability distribution function $P(\vec{v}_{\rm rel})$. Write down the expression for the probability distribution function $P(E_{\rm coll})$, where $E_{\rm coll}$ is the center-of-mass energy of collision between the two atoms.

Problem 5:

Consider N free particles confined inside a large cube of volume V, with periodic boundary conditions on the particles' wavefuntions. The density of states in 3-dimensional $\vec{k}$-space is given by

$$g(\vec{k}) = \frac{V}{(2\pi)^3}. \tag{0.0.3}$$

Take the above equation as given (don't need to prove) and show that if the particles are spin-1/2 fermions (i.e., 2 particles of opposite spins are allowed in each $\vec{k}$-state), then the Fermi surface has the radius

$$k_F = (3\pi^2 n)^{1/3}, \tag{0.0.4}$$

where $n = N/V$. In the non-relativistic case,

$$\epsilon(k) = \frac{\hbar^2 k^2}{2m}, \tag{0.0.5}$$

where m and ϵ are the particle mass and energy, respectively. Assume $T = 0$ and show that in that case the degeneracy pressure P of the fermions scales as

$$P \propto n^{5/3}. \tag{0.0.6}$$

Find the coefficient of proportionality. Repeat the same for the case of ultra-relativistic fermions, so that $\epsilon(k) = c\hbar k$. Show that

$$P \propto n^{4/3} \tag{0.0.7}$$

and find the coefficient of proportionality.

Problem 6:

(a, 5 points) Derive a partition function $Z(\beta, \omega)$ for a harmonic oscillator of proper frequency ω immersed in a heat bath of temperature T; here $\beta = (k_B T)^{-1}$ and k_B is the Boltzmann constant. Furthermore, find the mean energy

$$E = -\frac{\partial \log Z}{\partial \beta}. \tag{0.0.8}$$

Show that for $T \gg \hbar\omega/k_B$, $E = k_B T$.

(b, 5 points). The position operator $\hat{x}$ can be expressed in terms of the ladder (lowering and raising) operators a and $a^\dagger$ as follows:

$$\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}(a + a^\dagger). \tag{0.0.9}$$

Here m is the mass of the oscillator. Use this relation to show that the mean potential energy

$$U \equiv \frac{1}{2} m\omega^2 \langle \hat{x}^2 \rangle = \frac{1}{2} E. \tag{0.0.10}$$

Use this to estimate to order-of-magnitude the root-mean-square of the thermally fluctuating horizontal displacement of a $m = 40$kg pendulum attached to a 60cm long suspension fiber (these roughly correspond to the suspended mirrors of advanced LIGO). Assume the pendulum is in thermal equilibrium at room temperature $T = 300$K. You will need the following numbers: $k_B = 1.38 \times 10^{-23}$J/K, $\hbar \simeq 1.05 \times 10^{-34}$kg $\times$ m^2/s, and the free-fall acceleration on Earth is $g = 9.81$m /s^2.

Potentially Useful Equations and Definitions

Time-independent Perturbation Theory

$$E_n^{(1)} = \langle n^{(0)}|\hat{V}|n^{(0)}\rangle$$

$$|n^{(1)}\rangle = \sum_{m\neq n} \frac{|m^{(0)}\rangle \langle m^{(0)}|\hat{V}|n^{(0)}\rangle}{E_n^{(0)} - E_m^{(0)}}$$

$$E_n^{(2)} = \sum_{m\neq n} \frac{|\langle n^{(0)}|\hat{V}|m^{(0)}\rangle|^2}{E_n^{(0)} - E_m^{(0)}}$$

Harmonic Oscillator Equations

$$\hat{a}^\dagger = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{q} - \frac{i}{m\omega}\hat{p}\right)$$

$$\hat{a} = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{q} + \frac{i}{m\omega}\hat{p}\right)$$

$$\hat{a}\,|n\rangle = \sqrt{n}\,|n-1\rangle$$

$$\hat{a}^\dagger\,|n\rangle = \sqrt{n+1}\,|n+1\rangle$$

$$[\hat{a}, \hat{a}^\dagger] = 1$$

$$E_n = \hbar\omega\left(n + \frac{1}{2}\right)$$

Spin operators and Commutators

Useful Commutators:

$$[\hat{S}_x, \hat{S}_y] = i\hbar\hat{S}_z, \quad [\hat{S}_y, \hat{S}_z] = i\hbar\hat{S}_x, \quad [\hat{S}_z, \hat{S}_x] = i\hbar\hat{S}_y$$

Spin $1/2$

$$\hat{S}_x = \frac{\hbar}{2}\hat{\sigma}_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\hat{S}_y = \frac{\hbar}{2}\hat{\sigma}_y = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$\hat{S}_z = \frac{\hbar}{2}\hat{\sigma}_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Spin 1

$$\hat{S}_x = \frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\hat{S}_y = \frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}$$

$$\hat{S}_z = \hbar\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

Partition Function

$$Z(\beta) = \sum_n e^{-\beta E_n}$$

Useful Misc. Identities

Taylor Expansions:

$$e^{\alpha x} = \sum_n^{\infty} \frac{(\alpha x)^n}{n!} = 1 + \alpha x + \frac{(\alpha x)^2}{2!} + \frac{(\alpha x)^3}{3!} + \ldots$$
$$\sqrt{1+x} \approx 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} + \ldots$$

Commutators:

$$[A, BC] = [A, B]C + B[A, C]$$

Trigonometry:

$$sin(a+b) = sin(a)cos(b) + cos(a)sin(b)$$
$$cos(a+b) = cos(a)cos(b) - sin(a)sin(b)$$

Integrals:

$$\int dxe^{iax}x = \frac{1}{a^2}e^{iax}(1 - iax)$$
$$\int dxe^{iax}x^2 = \frac{1}{a^3}e^{iax}(-ia^2x^2 + 2ax + 2i)$$