Columbia University
Department of Physics
QUALIFYING EXAMINATION

Friday, January 18, 2019
2:00PM to 4:00PM
General Physics
Section 6. Various Topics

Two hours are permitted for the completion of this section of the examination. Choose 4 problems out of the 6 included in this section. (You will not earn extra credit by doing an additional problem). Apportion your time carefully.

Use separate answer booklet(s) for each question. Clearly mark on the answer booklet(s) which question you are answering (e.g., Section 6 (General Physics), Question 2, etc.).

Do **NOT** write your name on your answer booklets. Instead, clearly indicate your **Exam Letter Code**.

You may refer to the single handwritten note sheet on $8\frac{1}{2}$" $\times$ 11" paper (double-sided) you have prepared on Modern Physics. The note sheet cannot leave the exam room once the exam has begun. This note sheet must be handed in at the end of today's exam. Please include your Exam Letter Code on your note sheet. No other extraneous papers or books are permitted.

Simple calculators are permitted. However, the use of calculators for storing and/or recovering formulae or constants is NOT permitted.

Questions should be directed to the proctor.

Good Luck!

1. The spectrum of a diatomic molecule includes transitions between (1) electronic levels, between (2) vibrational levels, and between (3) rotational levels.

   (a) Given order of magnitude estimates of each of these transition energies (1)-(3) in terms of nuclear and electron masses and other fundamental constants.

   (b) Order the transitions (1)-(3) from lowest to highest energy.

   (c) At approximately what parts of the electromagnetic spectrum do these transitions correspond to?

2. What is the critical angle for total external reflection for photons of wavelength λ and frequency $\omega = 2\pi c/\lambda$ in vacuum, falling on a metal plate with electron density N? Assume that electrons in a metal are essentially free.

3. The radius of the event horizon of a (static) black hole of mass M may be estimated using Newtonian methods as the radius where the escape velocity reaches the speed of light. Objects crossing the event horizon disappear from view, but may experience large tidal forces while falling into the black hole.

   Estimate the mass of a black hole sufficiently large that a Sun-sized star would not be disrupted when crossing the event horizon. Express your answer as a multiple of the solar mass $M_\odot$. Take the radius of the Sun to be $R_\odot = 7 \times 10^5\,\mathrm{km}$, and use the fact that if it were to form a black hole its event horizon would be about $R_S \sim 3\,\mathrm{km}$.

4. A plane-parallel atmosphere in a uniform vertical gravitational field g is illuminated from above by a vertical radiation flux F $[\mathrm{erg} \cdot \mathrm{s}^{-1} \cdot \mathrm{cm}^{-2}]$. The atmosphere is isothermal, with a temperature T. It is composed of particles of mass m, which isotropically scatter radiation with cross section σ. The atmosphere is optically thin. The surface below it, at $z = 0$, absorbs the impinging radiation flux. The particle number density at $z = 0$ is n_0.

   (a) Find the vertical distribution of the atmosphere density $n(z)$ at $z > 0$.

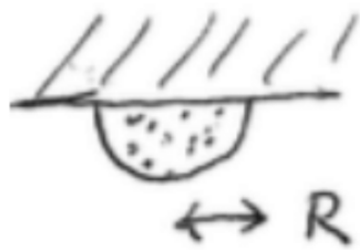
   (b) How will $n(z)$ change if the surface $z = 0$ reflects radiation rather than absorbs it?

5.

(a) The "surface tension" of water (σ) is the extra energy needed for a unit area increase of the surface between liquid water and an adjacent atmosphere. Use the information below to make a rough estimate of σ.

- Heat of vaporization of liquid water $= 2 \times 10^{10}$ ergs/cm^3
- Mass of a water molecule $= 3 \times 10^{-23}$ g
- Density of water $= 1$ g/cm^3

(b) Use σ (and any other natural constants you might need) to estimate the maximum radii of water droplet hemispheres, which may still be clinging to the outside flat horizontal bottom of a newly washed (but not yet dry) glass.



6.

(a) Estimate the mass of the sun from your knowledge of its distance and any other common facts.

(b) Estimate the radius of the sun from common sense observations.

(c) Estimate the mean interior pressure of the sun.

(d) Estimate the mean interior temperature of the sun.

(e) Estimate the mean energy of a proton in the sun.

(f) Estimate the proton-proton Coulomb barrier that must be exceeded in two protons are to initiate a fusion reaction.

(g) Considering the answers to (e) and (f), how is fusion able to proceed in the sun?

Question 1 Solution

Let m be the electron mass, M be the nuclear mass, and $a = \hbar/me^2$ be the Bohr radius. If you do not remember the Bohr radius, it can be derived from enforcing angular momentum quantization in the hydrogen atom.

(a) Typical electronic energy level transitions are given by

$$\Delta E_{\text{elec}} \sim \frac{e^2}{a} \sim \frac{me^4}{\hbar^2}.$$

Typical vibrational energy level transitions are given by

$$\Delta E_{\text{vib}} \sim \hbar\omega \sim \sqrt{\frac{e^8 m^3}{\hbar^4 M}} = \frac{me^4}{M}\left(\frac{m}{M}\right)^{1/2},$$

where the characteristic vibrational frequency is given by

$$\omega = \sqrt{\frac{V''(a)}{M}} \approx \sqrt{\frac{e^2}{a}\cdot\frac{1}{a^2}\cdot\frac{1}{M}} \approx \sqrt{\frac{e^8 m^3}{\hbar^4 M}},$$

where $V''(r) = d^2V/dr^2$, where $V = e^2/r$ is the characteristic Coulomb binding force.

Finally,

$$\Delta E_{\text{rot}} \sim \frac{J^2}{I} \sim \frac{\hbar^2}{Ma^2} \sim \frac{me^4}{\hbar^2}\left(\frac{m}{M}\right).$$

(b) We see that

$$\Delta E_{\text{elec}} : \Delta E_{\text{vib}} : \Delta E_{\text{rot}} \sim 1 : \left(\frac{m}{M}\right)^{1/2} : \frac{m}{M}.$$

In other words, the electronic transitions are the highest energy, followed by the vibrational and then the rotational.

(c) Typical electronic transitions in the visual orUV frequency range. Typical vibrational transitions are in the infrared range. Typical rotational transitions are in the radio or microwave range

Question 2 Solution

The critical angle is determined from the Snell's law

$$n_1 \cos\theta_1 = n_2 \cos\theta_2,$$

where the angles are measured with respect to the surface. We take $n_1 = 1$ and $n_2 = n$. For the critical angle, $\theta_2 = 0$, so $\cos\theta_c = n$. Now we calculate the index of refraction $n(\omega)$ in the metal. The equation of motion for a free electron is

$$m\frac{d^2x}{dt^2} = -eE.$$

For an AC electric field. $E = E_0 e^{-i\omega t}$ and $x = x_0 e^{-i\omega t}$, thus

$$x = \frac{e}{m\omega^2}E$$

The polarization of the metal as the dipole moment per unit volume is

$$P = -exN = -\frac{e^2N}{m\omega^2}E.$$

The polarizability is

$$\alpha = \frac{P}{E} = -\frac{e^2N}{m\omega^2}.$$

The dielectric function is

$$\epsilon = 1 + 4\pi\alpha \implies \epsilon = 1 - 4\pi\frac{e^2N}{m\omega^2}$$

Since $n^2 = \epsilon$, we find

$$\cos^2\theta_c = 1 - 4\pi\frac{e^2N}{m\omega^2} = 1 - \frac{\omega_p^2}{\omega^2},$$

where $\omega_p = 4\pi\dfrac{e^2N}{m}$ is the plasma frequency. It follows that $\sin\theta_c = \frac{\omega_p}{\omega}$. Thus for $\omega < \omega_p$ one finds total reflection at all angles.

Question 3 Solution

The wrong-headed Newtonian estimates for the event horizon magically gives the correct value:

$$\frac{1}{2}mc^2 = \frac{GMm}{R_H} \implies R_H = \frac{2GM}{c^2}.$$

Disruption occurs when the difference in force across a stellar radius evaluated at the horizon exceeds the gravitational force of attraction holding the star together:

$$\begin{aligned} F_{\text{tidal}} &\sim \frac{GM(M_\odot/2)}{(R_H - R_\odot/2)^2} - \frac{GM(M_\odot/2)}{(R_H + R_\odot/2)^2} \\ &\approx -R_\odot \frac{\partial}{\partial R}\frac{GM(M_\odot/2)}{R^2} \\ &= R_\odot \frac{2GM(M_\odot/2)}{R_H^3} \\ F_{\text{attract}} &\sim \frac{G(M_\odot/2)^2}{R_\odot^2}. \end{aligned}$$

Thus

$$F_{\text{tidal}} \leq F_{\text{attract}} \implies R_\odot \frac{2GM(M_\odot/2)}{R_H^3} \leq \frac{G(M_\odot/2)^2}{R_\odot^2} \implies 4\left(\frac{M}{M_\odot}\right) \leq \left(\frac{R_H}{R_\odot}\right)^3.$$

Noting that the event horizon is proportional to mass, and using the information provided for the Sun's Schwarzschild radius $R_S = 3$ km, this becomes

$$4\left(\frac{M}{M_\odot}\right) \leq \left(\frac{R_H}{R_S}\frac{R_S}{R_\odot}\right)^3 = \left(\frac{M}{M_\odot}\frac{R_S}{R_\odot}\right)^3 \implies \left(\frac{M}{M_\odot}\right)^2 \geq 4\left(\frac{R_\odot}{R_S}\right)^3.$$

Thus

$$\begin{aligned} \frac{M}{M_\odot} &\geq 2\left(\frac{R_\odot}{R_S}\right)^{3/2} \\ &= 2\left(\frac{7 \times 10^5 \text{ km}}{3 \text{ km}}\right)^{3/2} \\ &\approx 2 \times 10^8. \end{aligned}$$

Question 4 Solution

(a) The downward momentum flux of radiation is F/c. The scattered radiation is isotropic, and so has zero momentum. Hence each particle in the atmosphere receives from radiation downward momentum per unit time $F\sigma/c$. The net force applied to each particle is $f = -gm - F\sigma/c$. It is uniform and maybe described by potential $U(z) = -fz$. The atmosphere density is given by the Boltzmann distribution

$$n(z) = n_0 \exp\left(-\frac{U(z)}{T}\right), \qquad U(z) = \left(gm + \frac{F\sigma}{c}\right) z.$$

(b) If radiation is reflected by the surface then the radiation flux everywhere vanishes—the downward flux and upward (reflected) flux cancel each other. Hence radiation applies no force to the particles, and the atmosphere density is given by the usual expression

$$n(z) = n_0 \exp\left(\frac{-mgz}{T}\right).$$

Question 5 Solution

(a) Imagine the water molecules in a cubic arrangement with spacing a. Let E_B be the binding energy of deeply buried water molecule. This molecule will have six neighbors. A water molecule at the surface will only have five neighbors, so its binding energy will be $\frac{5}{6}E_B$. The the energy difference is $\Delta E = \frac{1}{6}E_B$.

Letting L, m and ρ be the heat of vaporization, the mass of a water molecule, and the density of water, respectively, we see by dimensional analysis that

$$L = \frac{E_B}{a^3}, \quad \sigma = \frac{E_B}{6a^2}, \quad \rho = \frac{m}{a^3}.$$

It follows that

$$\sigma = \frac{La}{6}.$$

After finding

$$a = \left(\frac{m}{\rho}\right)^{1/3} \approx 3 \times 10^{-8} \text{ cm},$$

we compute

$$\sigma \approx 100 \text{ erg/cm}^2.$$

(b) By dimensional analysis or direct application of the model, we find that the downward force is

$$F_g = \frac{2\pi}{3} R^3 \rho g,$$

while the upward force is proportional to the length of the boundary of the interface between the drop and the glass, i.e.,

$$F_T = 2\pi R\sigma.$$

Setting these equal and solving for R yields

$$R = \left(\frac{3\sigma}{\rho g}\right)^{1/2} \approx 0.5 \text{ cm}.$$

Question 6 Solution

(a) Since

$$\frac{GM_S}{r^2} = \omega^2 r \implies M_S = \frac{4\pi^2 r^3}{GT^2}.$$

Since $T = 1$ year $= 3.2 \times 10^7$ s and $r_S = 1.5 \times 10^{13}$ cm, we find

$$M_S = 2 \times 10^{33} \text{ g}.$$

(b) I can block out the sun with a penny about two feet from my eye. Thus

$$\frac{0.25 \text{ cm}}{60 \text{ cm}} = \frac{R_S}{r_S} \implies R_S = 7 \times 10^{10} \text{ cm}.$$

(c) We find

$$\langle P_S \rangle = \frac{2}{3}\frac{E_K}{V},$$

where E_K is the thermal kinetic energy. The virial theorem states

$$2E_K + E_G = 0,$$

where E_G is the gravitational binding energy. Thus

$$\langle P_S \rangle = \frac{1}{3}\frac{E_G}{V} = \frac{1}{3}\frac{fGM^2}{R}\frac{1}{\frac{4}{3}\pi R_S^3}$$

where f is a dimensionless constant of order one. Thus

$$\langle P_S \rangle \approx \frac{1}{4\pi}\frac{GM^2}{R_S^4} = 10^{15} \text{ dyne/cm}^2 = 10^{14} \text{ Pa}.$$

(d) We know that

$$\langle P_S \rangle = \frac{\langle \rho \rangle k_B T}{\overline{m}},$$

where $\overline{m} \approx m_p = 1.6 \times 10^{-24}$ g. We know that

$$\langle \rho \rangle = \frac{M_S}{\frac{4}{3}\pi R_S^3} \approx 1 \text{ g/cm}^2.$$

Thus

$$k_B T \approx 1.6 \times 10^{-9} \text{ erg} \approx 1 \text{ keV} \implies T \approx 10^7 \text{ K}.$$

(e) We find

$$\langle E_p \rangle = \frac{3}{2} k_B T = 1.5 \text{ keV}.$$

(f) For nuclear fusion, the protons need to get within the range of the strong nuclear force range, which is roughly the radius of the nucleus $r_N \approx 1$ fm. The Coulomb barrier at this distance is

$$E_{\text{Coulomb}} = \frac{e^2}{r_N} = \frac{e^2/\hbar c}{1 \text{ fm}} \hbar c = \frac{1}{137} \frac{197 \text{ MeV} \cdot \text{fm}}{1 \text{ fm}} = 1.4 \text{ MeV}.$$

(g) We see from the above calculations that

$$\langle E_p \rangle \ll E_{\text{coulomb}}.$$

Thus for fusion to happen there needs to be quantum mechanical tunneling through the barrier.