Columbia University
Department of Physics
QUALIFYING EXAMINATION

Friday, January 16, 2015
3:10PM to 5:10PM
General Physics (Part II)
Section 6.

Two hours are permitted for the completion of this section of the examination. Choose 4 problems out of the 6 included in this section. (You will not earn extra credit by doing an additional problem). Apportion your time carefully.

Use separate answer booklet(s) for each question. Clearly mark on the answer booklet(s) which question you are answering (e.g., Section 6 (General Physics), Question 2, etc.).

Do **NOT** write your name on your answer booklets. Instead, clearly indicate your **Exam Letter Code**.

You may refer to the single handwritten note sheet on $8\frac{1}{2}$" $\times$ 11" paper (double-sided) you have prepared on General Physics. The note sheet cannot leave the exam room once the exam has begun. This note sheet must be handed in at the end of today's exam. Please include your Exam Letter Code on your note sheet. No other extraneous papers or books are permitted.

Simple calculators are permitted. However, the use of calculators for storing and/or recovering formulae or constants is NOT permitted.

Questions should be directed to the proctor.

Good Luck!

1. Consider a cosmic-ray antiproton which is slowed down in a Silicon block, where it is eventually captured into a highly excited state of a Silicon atom. In such a case all of the Silicon electrons are quickly ejected by auto-ionization, leading to an exotic atom consisting only of a Silicon nucleus around which an antiproton is orbiting.

   (a) Find the energy spectrum of this exotic atom, and calculate the energy of the photon emitted when the antiproton makes a transition from the n=2 to n=1 state.

   (b) The Silicon block, with a proper voltage across it, can serve as a detector, since photons stopped in it can liberate electron-hole pairs which can be swept out of the silicon and collected in a charge sensitive preamplifier. Assume a photon of the energy of part a) is stopped in the silicon. Pick a reasonable value for the input capacitance of a unity gain preamplifier and estimate the maximum amplitude of the voltage pulse produced at its output.

   (c) Usually the Silicon block is cooled to a very low temperature. Explain qualitatively what advantage is gained by operating this Silicon detector at low temperatures.

   (d) Assume the antiproton enters the Silicon block with a kinetic energy of 100 MeV. The antiproton must be slowed down to effectively zero kinetic energy to be captured by a Silicon atom. Estimate the thickness of the Silicon block required to bring this antiproton to rest.

2.

(a) Estimate (for example, using the uncertainty relation) the localization length and the ground state energy for a particle of mass m in the potential $U(x) = Cx^s$, where $s = 2, 4, 6, \ldots$ (Numerical factors may be neglected.)

(b) Show that the result in a) correctly reproduces the harmonic oscillator case by appropriately choosing the parameters s and C.

3. You are on a long-distance flight having a cold drink and watching a romantic comedy, which inexplicably yet predictably makes you tear up a little. A teardrop ends up in your drink. What is approximately the increase in entropy of the tear+drink system due to the transfer of heat from tear to drink? By what factor does the corresponding microstate degeneracy of the system increase? (You may assume the specific heat capacity of the liquids involved equals the standard heat capacity of water, i.e. 75 J/K per mole, and you may pick your own estimates for tear size etc, but wait with plugging in specific numbers until the end of your computations.)

4. A black hole of mass M can carry nonzero electric charge Q and angular momentum L, but these are bounded by the inequalities

$$Q^2 \lesssim G \cdot M^2,\ L^2 \lesssim G^2 \cdot M^4,$$

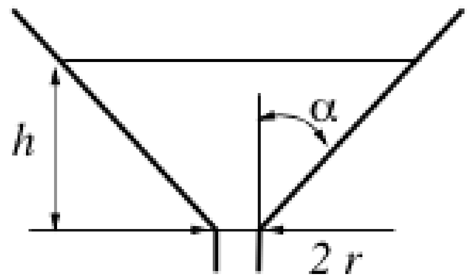
where G is Newton's constant, and we dropped numerical factors of order unity. Argue that if such inequalities were violated, the black hole could never have formed by gravitational collapse in the first place, because of too strong an electrostatic repulsion or centrifugal force. To this end, model the black hole right before the collapse as an homogeneous sphere of radius

$$R_S \sim GM \text{ (Schwarzschild radius)},$$

carrying constant mass density ρ_M and charge density ρ_Q, and rotating rigidly with some angular velocity ω. Ignore order-one numerical factors.

[Conventions: $c = 1$; Rationalized units for charge, e.g. fine structure constant is $\alpha = e^2/4\pi$.]

5. The stem of the conical funnel, see figure, is initially closed. The funnel is filled with an incompressible liquid (mass density ρ) up to the level h. At $t = 0$, the funnel is opened and the liquid begins to flow out.



(a) Find the flux, J, through the stem as a function of h.

(b) Find the velocity of the upper surface of the liquid $\mathrm{d}h/\mathrm{d}t$ as the function of the flux.

(c) Write down a differential equation whose solution would yield $h(t)$.

(d) Solve the differential equation to obtain an explicit form of $h(t)$.

6.

(a) Explain the concept of Umklapp scattering and describe the role this process plays in thermal conductivity due to phonons, and to electronic conductivity at low temperatures.

(b) Zero point motion is an important concept in quantum mechanics. Consider a phenomenon in solid state physics which involves zero point motion, and explain the role of zero point motion in this phenomenon. Provide an argument to estimate the magnitude of the zero point effect, and explain how one might measure this effect experimentally.

Exotic Atom soln ~~Ha~~ Hailey general

a.) use reduced mass $\frac{m_p M_{Si}}{m_p + M_{Si}} = \frac{28 \times 1}{28+1} \simeq m_p$

$mvr = n\hbar \; ; \; \frac{mv^2}{r} = \frac{Ze^2}{r^2} \; ; \; \frac{1}{2}mv^2 - \frac{Ze^2}{r} = E$

$\Rightarrow \Delta E_{n_i n_f} = -\frac{m Z^2 e^4}{2\hbar^2}\left(\frac{1}{n_i^2} - \frac{1}{n_f^2}\right)$

For $2 \to 1$ $\Delta E = \frac{3}{8} \frac{Z^2 m e^4}{\hbar^2} = \frac{3}{8} mc^2 \left(\frac{e^2}{\hbar c}\right)^2 Z^2$

$mc^2 \simeq 10^6$ keV; $e^2/\hbar c = \frac{1}{137}$ $Z = 14$ (Si)

$\Delta E \simeq 2000$ keV $\simeq 2$ MeV

b.) The energy to create e-h pair is ~ 3eV in a Si chip $\Rightarrow$ 600,000 e-h pairs.
A reasonable preamp would have C ~ few pF
so $V = q/C = \frac{6 \times 10^5 \times 1.6 \times 10^{-19} C}{2 \times 10^{-12} F} = \underline{50 \text{ mV}}$

c.) e-h pairs can be created by thermal agitation. Cooler $\Rightarrow$ less noise $\Rightarrow$ S/N better

d.) $dE/dx \simeq 2 \text{ MeV/g/cm}^2 \; \beta^{-2}$ Bethe-Bloch formula

$dE/dx = \frac{1/2\, m_p c^2}{E} \times 2 \text{ MeV/g/cm}^2$ $\quad \frac{1}{2} M_p c^2 \beta^2 = E$

$X = \int_0^{100\text{MeV}} \frac{dE}{(dE/dx)} = 10^{-3} \frac{\text{g/cm}^2}{\text{MeV}^2} \int_0^{100\text{MeV}} E\,dE = \underline{5 \text{ g/cm}^2}$ Ans

$\rho_{Si} \sim 2 \text{ g/cm}^3$ $\quad X = 2.5$ cm

SOLUTION: Localization length

a) $E = \langle K + U \rangle = \langle \frac{p^2}{2m} + C x^s \rangle$

$p \approx \frac{\hbar}{L}$

$\longrightarrow E \approx \frac{\hbar^2}{2mL^2} + CL^s$

Minimum energy:

$$\frac{dE}{dL} = -\frac{\hbar^2}{mL^3} + sCL^{s-1} = 0 \longrightarrow L = \left(\frac{\hbar^2}{msC}\right)^{\frac{1}{s+2}} \approx \boxed{\left(\frac{\hbar^2}{mC}\right)^{\frac{1}{s+2}}}$$

Substituting L into E, we find

$$\boxed{E \approx \left(\frac{\hbar^{2s} C^2}{m^s}\right)^{\frac{1}{s+2}}}$$

b) For a harmonic oscillator, set $\begin{cases} s = 2 \\ C = \frac{1}{2} m\omega^2 \end{cases}$

Then $L \approx \left(\frac{\hbar^2}{m^2\omega^2}\right)^{\frac{1}{4}} = \boxed{\sqrt{\frac{\hbar}{m\omega}}}$ ✓

$$E \approx \left(\frac{\hbar^4 \cancel{m^2} \omega^4}{\cancel{m^2}}\right)^{\frac{1}{4}} = \boxed{\hbar\omega}$$ ✓

3. You're on a long-distance flight having a cold drink and watching a romantic comedy, which inexplicably yet predictably makes you tear up a little. A teardrop ends up in your drink. What is approximately the increase in entropy of the tear+drink system due to the transfer of heat from tear to drink? By what factor does the corresponding microstate degeneracy of the system increase? (You may assume the specific heat capacity of the liquids involved equals the standard heat capacity of water, i.e. 75 J/K per mole, and you may pick your own estimates for tear size etc, but wait with plugging in specific numbers until the end of your computations.)

The total entropy change is

$$\Delta S = \Delta S_{\text{drink}} + \Delta S_{\text{tear}} = \int \frac{dU}{T}\bigg|_{\text{drink}} + \int \frac{dU}{T}\bigg|_{\text{tear}} . \tag{1}$$

Since the tear is much smaller than the drink, the temperature of the drink will remain virtually constant at $T = T_{\text{drink}}$, so the first term is to a good approximation:

$$\Delta S_{\text{drink}} = \frac{\Delta U_{\text{drink}}}{T_{\text{drink}}} , \quad \text{with} \quad \Delta U_{\text{drink}} = -\Delta U_{\text{tear}} = C_{\text{tear}}(T_{\text{tear}} - T_{\text{drink}}) . \tag{2}$$

Here C_{tear} is the heat capacity of the tear, which is essentially constant over this temperature range. The second term ΔS_{tear} is more complicated because the temperature of the tear decreases by a significant amount during this process:

$$\Delta S_{\text{tear}} = \int_{T_{\text{tear}}}^{T_{\text{drink}}} \frac{C_{\text{tear}} dT}{T} = C_{\text{tear}} \log\left(\frac{T_{\text{drink}}}{T_{\text{tear}}}\right) . \tag{3}$$

Putting the two together, we get, in the tear $\ll$ drink approximation:

$$\Delta S = C_{\text{tear}} \left[\frac{T_{\text{tear}} - T_{\text{drink}}}{T_{\text{drink}}} - \log\left(\frac{T_{\text{tear}}}{T_{\text{drink}}}\right)\right] . \tag{4}$$

Let us assume the drink is chilled cold and the tear straight from the tear duct, say $T_{\text{drink}} = 275\,\text{K}$, $T_{\text{tear}} = 310\,\text{K}$. One tear weighs about 1/20 of a gram, and 1 gram water is 1/18 mole, so the heat capacity of the tear is $C_{\text{tear}} = 0.21$ J/K, and

$$\boxed{\Delta S = 1.6 \times 10^{-3}\,\text{J/K}} . \tag{5}$$

The microstate degeneracy thus increases by a factor

$$\boxed{e^{\Delta S/k} = e^{1.1 \times 10^{20}}} . \tag{6}$$

1 General: Extremality bounds for black holes

A black hole of mass M can carry nonzero electric charge Q and angular momentum L, but these are bounded by the inequalities

$$Q^2 \lesssim G \cdot M^2 \, , \qquad L^2 \lesssim G^2 \cdot M^4 \, , \tag{1}$$

where G is Newton's constant, and we dropped numerical factors of order unity.

Argue that if such inequalities were violated, the black hole could never have formed by gravitational collapse in the first place, because of too strong an electrostatic repulsion or centrifugal force. To this end, model the black hole right before the collapse as an homogeneous sphere of radius

$$R_S \sim GM \qquad\qquad \text{(Schwarzschild radius)} \, , \tag{2}$$

carrying constant mass density ρ_M and charge density ρ_Q, and rotating rigidly with some angular velocity ω. Ignore order-one numerical factors.

[Conventions: $c = 1$; Rationalized units for charge, e.g. fine structure constant is $\alpha = e^2/4\pi$.]

Solution

Ignoring 2's and π's, for an homogeneous rotating sphere we have the relations

$$M \sim \rho_M R_S^3 \, , \qquad Q \sim \rho_Q R_S^3 \, , \qquad L \sim M\omega R_S^2 \, . \tag{3}$$

The gravitational, electrostatic, and centrifugal forces acting on an infinitesimal volume element of mass dm and charge dq at a distance r from the center are

$$dF_g \sim G\frac{\rho_M r^3 \, dm}{r^2} \sim G\rho_M r \cdot dm \, , \qquad dF_e \sim \frac{\rho_Q r^3 \, dq}{r^2} \sim \rho_Q r \cdot dq \, , \qquad dF_c \sim \omega^2 r \cdot dm \tag{4}$$

(in the last one there should be an order-one trigonometric factor that encodes the fact the the centrifugal force depends on the distance from the rotation axis rather than from the center.) The comparison among them is r-independent (they all feature the same r-dependence), and so demanding that gravity win over the other two forces we simply get (upon using $dq/dm = \rho_Q/\rho_M$)

$$\rho_Q^2 \lesssim G \cdot \rho_M^2 \, , \qquad \omega^2 \lesssim G \cdot \rho_M \tag{5}$$

Multiplying the former by the volume of the sphere, the latter by $M^2 R_S^4$, and using eqs. (2), (3), we get the desired bounds in eq. (1).

Solution:

1.
$$J = \rho v \pi r^2; \quad v = \sqrt{2gh};$$

2.
$$-\rho \frac{d}{dt}\left(\frac{\pi h^3 \tan^2\alpha}{3}\right) = J;$$

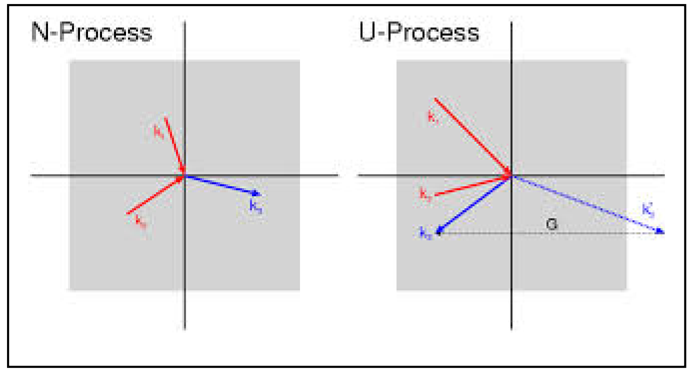
3.
$$\rho \frac{d}{dt}\left(\frac{\pi h^3 \tan^2\alpha}{3}\right) = -\rho v \pi r^2 = -\rho \pi r^2 \sqrt{2gh};$$
$$\dot{h} h^{3/2} = -\frac{r^2}{\tan^2\alpha}\sqrt{2g};$$

4.
$$h(t) = \left(h(t=0)^{5/2} - t\frac{5r^2\sqrt{2g}}{2\tan^2\alpha}\right)^{2/5}$$

Uemura

Sec 6 -6

a) Explain the concept of Umklapp scattering and describe the role this process plays in thermal conductivity due to phonons, and to electronic conductivity at low temperatures.
b) Zero point motion is an important concept in quantum mechanics. Consider a phenomenon in solid state physics which involves zero point motion, and explain the role of zero point motion in this phenomenon. Provide an argument to estimate the magnitude of the zero point effect, and explain how one might measure this effect experimentally.

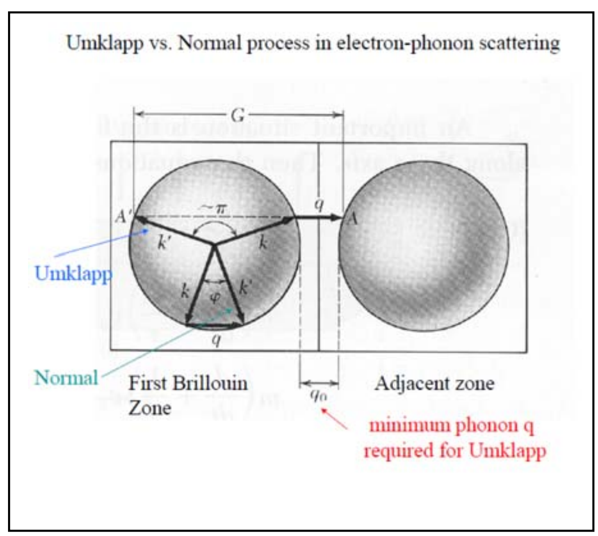


The Umklapp Process happens in scattering of phonons with a rather large momentum. As shown in the right figure, when added momentum of two phonons goes beyond the first Brillouin zone, the momentum is shifted by the Crystal momentum to be within the first Brillouin zone. In this case, momentum does not conserve, and there is net momentum deposition in the scattering process. For phonons with smaller momentum, called normal (n) process shown in the left figure, momentum is conserved.

The n-process does not contribute to change of net energy and momentum distribution, and there is no effect of scattering in heat transport. Energy flow by phonons proceed as if there were no scattering. In contrast, the Umklapp process contributes to the thermal resistance.

Thermal conductivity is generally given by a product of the specific heat C, the velocity of phonon (or other heat carrier) v, and the phonon mean free path *l*. At low temperatures, most of thermally excited phonons have small momentum, and only the “n process” happens in scattering. This does not influence the mean free path for thermal conductivity, and *l* is usually determined by imperfection or boundary of specimen. In that case, thermal conductivity increases with temperature, proportional to the specific heat, because phonon velocity is nearly independent on T and the mean free path is determined by geometry / imperfection.

At higher temperatures, sufficient number of Umklapp process can occur, and the mean free path *l* is inversely proportional to the number of phonon collision events, which is proportional to kT. Here, the specific heat shows saturation, and the phonon velocity is nearly independent of temperature. Therefore, the thermal conductivity is proportional to 1/kT.



The above mentioned case is for insulators where thermal conductivity is determined only by phonons. For metals, thermal conductivity and electric resistivity are both determined by scattering of electrons near Fermi surface by phonons. Here again, if phonons have sufficient momentum, the U-process can occur, and current / heat conduction can be disturbed effectively. The mean free path of electrons become inversely proportional to kT at high temperatures (called Umklapp temperature) above which there are sufficient high momentum phonons to scatter an electron on the Fermi surface to the Fermi surface of adjacent Brillouin zone (see figure). For lower T, electrons can be scattered only to Fermi surface of a given Brillouin zone, which occurs with a rather small momentum change. Therefore, the n-process is less influential to change the current path and resistivity shows higher power in T in the low temperature region.

Uemura
Sec 6 -6
Solution

Examples of the effect of zero-point motion in solid state physics.

(1) Liquid He, and He atom vibration in solid He

He atoms attract with each other via van-der-Waals force, which gives Lenard Jones interatomic potential. Since He atom is very light, the zero point motion energy prevents formation of solid in ambient pressure. With a small pressure, He solidifies into an HCP crystal. Even in the solid phase, zero-point motion is very large, and the atomic position has a large spread. This “zero-point vibration” of He atoms in HCP solid He can be measured by Form Factor of x-ray or neutron scattering.

(2) Fermi energy of a metal can be estimated in the following way:

(a) distribute every electron into a different cube. The dimension of this cube is the interelectron distance d.

(b) the uncertainty relationship will give a corresponding momentum for the distance d, which will lead to an energy due to confinement of an electron to a small cube …. i.e., a kind of zero point vibration energy for this given electron.

Carrier density n,
d ~ n^{-1/3}
p ~ (hbar)*n^{1/3}
effective mass m*
kinetic energy per electron p^{2}/2m* = (hbar)^{2}/2 * [n^{2/3} / m*]

Actual Fermi Energy (hbar)^{2} (3\pi^{2})^{2/3}/2 * [n^{2/3} / m*]
Average energy per electron = 0.6 * Fermi Energy

These two calculations are within a factor of ~ 3 or so. This implies that Fermi energy is actually a kinetic energy due to zero point motion. Since electron is a fermion, the simple model required to “confine” each electron to its own territory (the cube) to avoid overlap.

Fermi energy can be measured in various ways. Most directly by Angle Resolved Photo Emission Spectroscopy (ARPES) which allows entire mapping of energy dispersion relation of electrons.