Columbia University
Department of Physics
QUALIFYING EXAMINATION

Wednesday, January 14, 2015
3:10PM to 5:10PM
Modern Physics
Section 4. Relativity and Applied Quantum Mechanics

Two hours are permitted for the completion of this section of the examination. Choose 4 problems out of the 5 included in this section. (You will not earn extra credit by doing an additional problem). Apportion your time carefully.

Use separate answer booklet(s) for each question. Clearly mark on the answer booklet(s) which question you are answering (e.g., Section 4 (Relativity and Applied QM), Question 2, etc.).

Do **NOT** write your name on your answer booklets. Instead, clearly indicate your **Exam Letter Code**.

You may refer to the single handwritten note sheet on $8\frac{1}{2}$" $\times$ 11" paper (double-sided) you have prepared on Modern Physics. The note sheet cannot leave the exam room once the exam has begun. This note sheet must be handed in at the end of today's exam. Please include your Exam Letter Code on your note sheet. No other extraneous papers or books are permitted.

Simple calculators are permitted. However, the use of calculators for storing and/or recovering formulae or constants is NOT permitted.

Questions should be directed to the proctor.

Good Luck!

1. A particle of charge q and mass m is moving through the lab with a velocity v (you may not assume v is small compared to the speed of light) in the x-direction when it encounters a constant electric field of magnitude E in the y-direction. Find an equation for the subsequent motion of the particle $y(x)$.

2. Two electrons are confined by a parabolic potential and interact with each other via Coulomb interaction:

$$\hat{H} = -\frac{\hbar^2}{2m}(\vec{\nabla}_1^2 + \vec{\nabla}_2^2) + \frac{m\omega_0^2}{2}(\vec{r}_1^{\,2} + \vec{r}_2^{\,2}) + \frac{e^2}{|\vec{r}_1 - \vec{r}_2|}$$

where m is the electron mass, e is the electron charge and ω_0 is the frequency of oscillation of one electron in the parabolic potential.

(a) Separate the motion of the center of mass and the relative motion of the two electrons.

(b) Find the energy levels corresponding to the motion of the center of mass.

(c) Assuming $\gamma \equiv x_0/a_B \gg 1$, find the low-lying energy levels corresponding to the relative motion of two electrons. Here, $a_B = \hbar^2/(me^2)$ is the Bohr radius and $x_0 = \sqrt{2\hbar/(m\omega_0)}$ is the amplitude of the zero-point motion of the harmonic oscillator. Give your answer up to and including terms of the order $1/\gamma^{2/3}$.

(d) Find the degeneracy of the two lowest eigenvalues of the system at $\gamma \gg 1$ (don't forget about spin).

3. Recall the twin paradox setup. Let us call the twins Mary and Jane. Mary travels to a distant star at a speed v while Jane stays put on the Earth (v is with respect to the Earth; $v > 0$ by definition). When Mary reaches the distant star (and instantly commences her return journey to the Earth at the same speed v), Jane decides to travel to meet Mary at some intermediate point between the Earth and the star. Jane commences her trip at precisely the same moment Mary commences her return journey ('the same moment' according to the Earth's rest frame). Jane chooses to travel at a constant speed such that by the time she intercepts Mary, she would be at exactly the same age as Mary. This is only possible if Mary's speed v is not too great. What is that limiting speed?

4. A proton and a neutron each with same mass M and spin $\frac{1}{2}$ are bound by spin, $\vec{\sigma}_p/2$, $\vec{\sigma}_n/2$, $\vec{S} = (\vec{\sigma}_p + \vec{\sigma}_n)/2$, and orbital momentum, $\vec{L}$, dependent potentials of the form

$$V = V_1(r) + V_2(r)(\vec{\sigma}_p \cdot \vec{\sigma}_n) + V_3(r)(\vec{L} \cdot \vec{S})$$

where $\vec{\sigma}_p^i/2$ and $\vec{\sigma}_n^i/2$ generate spin rotations of the p and n separately.

(a) Is the Hamiltonian invariant under separate rotations of spin and orbital angular momenta? Show that it is invariant under the combined rotations generated by the total angular momentum $\vec{J} = \vec{L} + \vec{S}$ operators.

(b) Which combinations of the $(\vec{J}, \vec{L}, \vec{S})$ operators form complete compatible observables?

(c) Chose the basis that diagonalizes the potential and write down the full Hamiltonian including the kinetic energy term in that basis.

(d) Given the fact the deuteron has total $J = 1$ and positive parity, what is the most general dependence of its wavefunction on the spins of the proton and neutron and the direction of their relative coordinate?

5. An observer moving at a velocity v with respect to a blackbody radiation field of temperature T observes radiation that depends on the angle θ' measured with respect to the direction of motion.

   (a) Show that the observed radiation intensity $I_\nu(\theta')$ in every direction remains of the blackbody form. Calculate its temperature as a function of v and θ'.

   Useful information: (1) I_ν/ν^3 is a relativistic invariant; (2) the blackbody intensity of a source at temperature T is $I_\nu^{BB} = (2h\nu^3/c^2)(e^{h\nu/kT} - 1)^{-1}$.

   (b) The isotropy of the 2.7 K universal blackbody radiation at $\lambda = 3\,\mathrm{cm}$ has been established to about one part in 10^3. What is the maximum velocity that the earth can have with respect to the frame in which this radiation is isotropic?

   Isotropy is measured by the ratio $(I_{\max} - I_{\min})/(I_{\max} + I_{\min})$, where $I_{\min}$ and $I_{\max}$ are the minimum and maximum intensity, respectively. Answers valid to a few tens of percent accuracy are sufficient.

Hailey SR

Solution $\frac{dp^\mu}{d\tau} = e F^\mu{}_\nu u^\nu$ $u^\nu = (\gamma, \gamma v, 0, 0)$

$c = 1$

using $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$ $E = E\hat{y}$

And $g_{\mu\nu} = (-1, 1, 1, 1)$ one can quickly show that only $F^t{}_y = F^y{}_t = E$ Are non-zero.

$dp^x/d\tau = 0 \Rightarrow mu^x = \text{const} = \gamma m v$

$u^x = \gamma v \Rightarrow x = \gamma v \tau$

$dp^t/d\tau = e F^t{}_y u^y$

$dp^y/d\tau = \frac{e}{m} F^y{}_t u^t = \frac{e}{m} E u^t$

$\Rightarrow du^t/d\tau = e/m \, E u^y ; \frac{du^y}{d\tau} = \frac{e}{m} E u^t$

$d^2u^y/d\tau^2 = \left(\frac{e}{m}E\right)^2 u^y ; \quad u^y = \gamma \sinh \frac{eE}{m}\tau$

[using $u^y(0) = 0 ; \left.\frac{du^y}{d\tau}\right|_{\tau=0} = \gamma \frac{eE}{m}$]

$\tau = \frac{x}{\gamma v} \Rightarrow u^y = \frac{dy}{d\tau} = \gamma \sinh \frac{eE}{m}\tau$

$y(x) = \frac{m}{qE}\gamma\left[\cosh \frac{eE}{m}\frac{x}{v\gamma} - 1\right]$ using

$y(0) = 0$ And $\tau = x/\gamma v$

Hailey 4-1

Hailey
Sec 4 -1
Solution
(Revised)

Alternate soln without Tensors

$\frac{dP_y}{dt} = qE \; ; \; \frac{dP_x}{dt} = 0$ $\qquad \vec{P} = \gamma m \vec{v}$

set $c = 1$

$P_x = \text{constant} = P_0 = \gamma m v$

$P_y = qEt \; ; \quad \mathcal{E} = \sqrt{p^2 + m^2} = \sqrt{m^2 + P_0^2 + (qEt)^2}$

$\mathcal{E} = \sqrt{\mathcal{E}_0^2 + (qEt)^2} \; ; \quad \mathcal{E}_0 \equiv \sqrt{P_0^2 + m^2}$

$\vec{v} = \vec{P}/\mathcal{E} \Rightarrow \frac{dy}{dt} = \frac{qEt}{\sqrt{\mathcal{E}_0^2 + (qEt)^2}}$

$dx/dt = \frac{P_0}{\sqrt{\mathcal{E}_0^2 + (qEt)^2}}$

$\Rightarrow \quad y(t) = \left[\frac{1}{Eq} \sqrt{\mathcal{E}_0^2 + q^2E^2t^2} - \frac{\mathcal{E}_0}{Eq} \right] \qquad y(0) = 0$

$x(t) = \frac{P_0}{qE} \sinh^{-1}\left(\frac{qEt}{\mathcal{E}_0}\right)$ using substitution

$qEt = \mathcal{E}_0 \sinh w$ to solve integral in x

putting in $t = \frac{\mathcal{E}_0}{qE} \sinh \frac{qEx}{P_0}$

into $y(t)$ gives

$y(x) = \frac{\mathcal{E}_0}{qE} \left[\cosh\left(\frac{qEx}{P_0}\right) - 1 \right]$

$y(x) = \frac{\gamma m}{qE} \left[\cosh\left(\frac{qEx}{\gamma m v}\right) - 1 \right]$

Solution

1. Substituting wave-function

$$\Psi(\vec{r}_1, \vec{r}_2) = \psi\left(\frac{\vec{r}_1 + \vec{r}_2}{2}\right)\phi\left(\vec{r}_1 - \vec{r}_2\right) \tag{2}$$

into the Schrödinger equation with the Hamiltonian (1), we obtain

$$\left[-\frac{\hbar^2\vec{\nabla}_R^2}{2(2m)} + \frac{(2m)\omega_0^2\vec{R}^2}{2}\right]\psi\left(\vec{R}\right) = E_{cm}\psi\left(\vec{R}\right), \tag{3}$$

$$\left[-\frac{\hbar^2\vec{\nabla}_r^2}{m} + \frac{m\omega_0^2\vec{r}^{\,2}}{4} + \frac{e^2}{|\vec{r}|}\right]\phi\left(\vec{r}\right) = E_r\phi\left(\vec{r}\right), \tag{4}$$

the energy of the system is given by

$$E = E_{cm} + E_r. \tag{5}$$

2. Hamiltonian from Eq. (3) is nothing but the three-dimensional harmonic oscillator with eigenenergies

$$E_{cm} = \hbar\omega_0\left(\frac{3}{2} + i_x + i_y + i_z\right); \quad i_{x,y,z} = 0, 1, 2, \ldots \tag{6}$$

3. Hamiltonian (4) is rotationally invariant and therefore the angular and relative motions can be separated:

$$\phi\left(\vec{r}\right) = \frac{\chi(|\vec{r}|/x_0)}{|\vec{r}|}Y_{l,\,l_z}\left(\frac{\vec{r}}{|\vec{r}|}\right), \tag{7}$$

where Y is the spherical function and $l,\ l_z$ are the eigenvalues of the angular momentum and its z- component respectively. Substituting Eq. (7) into Eq. (4), we find

$$\left[-\frac{d^2}{2dx^2} + \frac{l(l+1)}{2x^2} + U(x)\right]\chi(x) = \epsilon\chi\left(x\right), \quad U(x) = \frac{x^2}{2} + \frac{\gamma}{x}, \quad E_r = \hbar\omega_0\epsilon, \tag{8}$$

with the boundary condition

$$\chi(0) = 0. \tag{9}$$

At $\gamma \gg 1$ the wavefunction is localized near the minima of the potential $U(x)$, $x_{min} = \gamma^{1/3}$ so that it is sufficient to expand this potential in the vicinity of this minima:

$$\left[-\frac{d^2}{2dx^2} + \frac{l(l+1)}{2\gamma^{2/3}} + \frac{3}{2}\gamma^{2/3} + \frac{3}{2}\left(x - \gamma^{1/3}\right)^2\right]\chi(x) = \epsilon\chi(x), \qquad (10)$$

The eigenfunction decays like $\exp\left(-(x-\gamma^{1/3})^2\right)$, so that the boundary condition (9) can be neglected with the exponential accuracy, and Eq. (10) reduced to the one for the harmonic oscillator with the frequency $\sqrt{3}$. We obtain with the required accuracy

$$\epsilon = \frac{3}{2}\gamma^{2/3} + \sqrt{3}\left(n + \frac{1}{2}\right) + \frac{l(l+1)}{2\gamma^{2/3}}, \quad n, l = 0, 1, 2, \ldots$$

which gives

$$E_r = \hbar\omega_0\left(\frac{3\gamma^{2/3}}{2} + \sqrt{3}\left(n + \frac{1}{2}\right) + \frac{l(l+1)}{2\gamma^{2/3}}\right). \qquad (11)$$

The total energies (5) are given by

$$E = \hbar\omega_0\left(\frac{3\gamma^{2/3}}{2} + \sqrt{3}\left(n_1 + \frac{1}{2}\right) + \left(n_2 + \frac{1}{2}\right) + \frac{l(l+1)}{2\gamma^{2/3}}\right), \qquad (12)$$

where $n_1, n_2, l = 0, 1, 2, \ldots$.

4. Ground state is given by $n_{1,2} = 0$, $l = 0$. Therefore the total spin $S = 0$, and the state is not degenerate.

   First excited state is $n_{1,2} = 0$, $l = 1$. The orbital function is odd, thus, $S = 1$. We have 3-fold degeneracy for the orbital motion and 3-fold degeneracy for spin. The total degeneracy of the state is 9.

5. Total momentum $\vec{J} = \vec{L} + \vec{S}$ is conserved. Thus, the ground state is not affected, and the first excited state is split so that the eigenfunctions are eignefunctions of $\vec{J}^2$. We obtain $J = 0$ (non-degenerate), $J = 1$ (3-fold degenerate) and $J = 2$ (five fold degenerate).

This is for the Quals to be held in January 2015.

Special relativity problem. Recall the twin paradox setup. Let us call the twins Mary and Jane. Mary travels to a distant star at a speed v while Jane stays put on the Earth (v is with respect to the Earth; $v > 0$ by definition). When Mary reaches the distant star (and instantly commences her return journey to the Earth at the same speed v), Jane decides to travel to meet Mary at some intermediate point between the Earth and the star. Jane commences her trip at precisely the same moment Mary commences her return journey ('the same moment' according to the Earth's rest frame). Jane chooses to travel at a constant speed such that by the time she intercepts Mary, she would be at exactly the same age as Mary. This is only possible if Mary's speed v is not too great. What is that limiting speed? Sec 4 - 3

Solution (Lam Hui). Let us refer to Jane's speed as u. The distance between the Earth and the star is defined as unity. The constraint of equal elapsed proper time for Jane and Mary is

$$\frac{1}{v} + \frac{1}{u+v}\sqrt{1-u^2} = \left(\frac{1}{v} + \frac{1}{u+v}\right)\sqrt{1-v^2} \tag{1}$$

The limiting speed can be worked out by setting $u = 1$, the greatest possible speed for Jane. One finds $v = 1/\sqrt{2}$.

Quals 2015: Applied QM (M.Gyulassy)

1. A proton and a neutron each with same mass M and spin $\frac{1}{2}$ are bound by a spin, $\vec{\sigma}_p/2, \vec{\sigma}_n/2, \vec{S} = (\vec{\sigma}_p + \vec{\sigma}_n)/2$, and orbital angular momentum, $\vec{L}$, dependent potentials of the form

$$V = V_1(r) + V_2(r)(\vec{\sigma}_p \cdot \vec{\sigma}_n) + V_3(r)(\vec{L} \cdot \vec{S})$$

where $\vec{\sigma}_p^i/2$ and $\vec{\sigma}_n^j/2$ generate spin rotations of the p and n separately.

a) Is the Hamiltonian ~~is~~ invariant under seperate rotations of spin and orbital angular momenta? Show that it is invariant under the combine rotations generated by the total angular momentum $\vec{J} = \vec{L} + \vec{S}$ operators.

b) Which combinations of the $(\vec{J}, \vec{L}, \vec{S})$ operators form complete compatible observables?

c) Chose the basis that diagonalizes the potential and write down the full Hamiltonian including the kinetic energy term in that basis.

d) Given the fact the the deuteron has total $J = 1$ and positive parity, what is the most general form of its wavefunction?

1. Solution

a) V_2 term requires both spins to rotate the same way at once to maintain their inner product, the V_3 term requires orbital to rotate the same way as total spin. On the other hand $[J^i, (\vec{\sigma}_p \cdot \vec{\sigma}_n) = 0$ since $[\sigma_p^i, (\vec{\sigma}_p \cdot \vec{\sigma}_n)] = i\epsilon^{ijk}\sigma_p^k\sigma_n^j$ while $[\sigma_n^i, (\vec{\sigma}_p \cdot \vec{\sigma}_n)] = i\epsilon^{ijk}\sigma_p^j\sigma_n^k$ so those terms cancel due to antisymmetry of Levi-Civita tensor ϵ_{ijk}, and $L_i = \epsilon_{ijk}x_jp_k$ commutes with any function $|\vec{r}|$ since $\epsilon_{ijk}x_jx_k = 0$ and also with both intrinsic spin operators. Similar cancellation occurs for spin orbit term. So the Ham is invariant under 3D rotations generated by J_i.

b) The uncoupled $|L, m_L, 1/2, m_p, 1/2, m_n >$ basis is an invariant basis of the set of commuting operators $L^2, L_z, s_p^2, s_{pz}, s_n^2, s_{nz}$. However the most useful basis in this problem is the coupled angular basis $|J, L, S, M_z = m_L + m_S >$ connected to the uncoupled basis through Clebsch coeff.

c) rewrite V using the coupled basis $(\vec{\sigma}_p \cdot \vec{\sigma}_n) \to 2(S(S+1) - 3/2)$ and $(\vec{L} \cdot \vec{S}) \to \frac{1}{2}(J(J+1) - L(L+1) - S(S+1))$ and recall that the reduced mass of the equal mass system is $M/2$, so that in terms of the relative coordinate r, the radial hamiltonian in the J, L, S invariant subspace is

$$\begin{aligned} H_{JLS} &= -\frac{\hbar^2}{Mr}\frac{d^2}{dr^2}r + \frac{\hbar^2}{M}\frac{L(L+1)}{r^2} + V_1(r) \\ &\quad +2(S(S+1) - 3/2)V_2(r) + \tfrac{1}{2}(J(J+1) - L(L+1) - S(S+1))V_3(r) \qquad (1) \end{aligned}$$

d) $S = 0, 1$ are only possibilities, $L = 1$ is ruled out by parity, but $L = 0$ and $L = 2$ could mix if additional tensor perturbations (e.g. of the form $V(r)(\vec{\sigma}_p \cdot \vec{r})(\vec{\sigma}_n \cdot \vec{r})$) exist. Thus, $\psi = u_{L=0}(r)\chi_{J=1,L=0,S=1,M}(\hat{n}) + u_{L=2}(r)\chi_{J=1,L=2,S=1,M}(\hat{n}$ where the $\chi_{JLSM} = \sum_{m,m'} Y_{Lm}(\hat{n})|S, m' >< LSmm'|JM >$ are the usual spherical harmonics and spinors combined with Clebsch Gordon coef.

Solution:

(a) Define the primed frame as that of the observer and the unprimed frame as that of the stationary blackbody radiation field of temperature T. From the relativistic invariance of $I_\nu \nu^3$ we have

$$\frac{I_\nu}{\nu^3} = \frac{I'_{\nu'}}{\nu'^3}. \tag{9}$$

Substituting in the blackbody formula for I_ν we get the intensity seen by the moving observer

$$I'_{\nu'} = \frac{2h\nu'^3}{c^2}\left(e^{h\nu/kT} - 1\right)^{-1} \tag{10}$$

Likewise, the frequency in the rest frame of the blackbody is related to that of the moving observer by the Doppler formula

$$\nu = \nu'\gamma\left(1 - \frac{v}{c}\cos\theta'\right), \tag{11}$$

where $\gamma = (1 - v^2/c^2)^{-1/2}$ (the sign of $\cos\theta'$ is negative of the conventional case because radiation propagates in the direction opposite to the viewing angle). Substituting into equation (12) we have

$$I'_{\nu'} = \frac{2h\nu'^3}{c^2}\left(e^{h\nu'\gamma(1-v\cos\theta'/c)/kT} - 1\right)^{-1} \tag{12}$$

The intensity seen by the moving observer can thus be written in the BB form $I'_{\nu'} = \frac{2h\nu'^3}{c^2}\left(e^{h\nu'/kT'} - 1\right)^{-1}$ provided one defines a new temperature

$$T' \equiv \frac{T}{\gamma(1 - v\cos\theta'/c)} = T\frac{\sqrt{1 - v^2/c^2}}{1 - v\cos\theta'/c} \tag{13}$$

(b) A small anisotropy requires a slow velocity $v \ll c$. Expanding for $v/c \ll 1$, we obtain $T' \approx T(1+v\cos\theta'/c)$, so that the maximum and minimum temperatures are $T'_{\rm max} \approx T(1+\beta)$ and $T'_{\rm min} \approx T(1-\beta)$, where $\beta = v/c$. We can thus approximate

$$I_{\rm max} \propto \left(e^{h\nu'/kT'_{\rm max}} - 1\right)^{-1} \propto 1 - \beta \tag{14}$$

$$I_{\rm min} \propto \left(e^{h\nu'/kT'_{\rm min}} - 1\right)^{-1} \propto 1 - \beta \tag{15}$$

for $h\nu/kT \sim 0.2 \ll 1$. Thus

$$\frac{I_{\rm max} - I_{\rm min}}{I_{\rm max} + I_{\rm min}} \approx v/c \lesssim 10^{-3} \tag{16}$$

so that $v \lesssim 10^{-3}c \sim 300$ km s^{-1}.

Solution

(a) The heat in per unit mass is:

$$\Delta Q = \frac{1}{2} C (T_1 - T) - \frac{1}{2} C (T - T_2)$$

For steady state,

$$\frac{v^2}{2} = \Delta Q$$

$$\Rightarrow \quad v = \sqrt{C(T_1 + T_2 - 2T)}$$

(b) Entropy increase is positive

$$\Delta S = \frac{1}{2} C \ln \frac{T}{T_1} + \frac{1}{2} C \ln \frac{T}{T_2} \geq 0$$

$$= \frac{1}{2} C \ln \frac{T^2}{T_1 T_2} \geq 0$$

$$T^2 \geq T_1 T_2 \quad \text{or} \quad T \geq \sqrt{T_1 T_2}$$

$$V_{MAX} = \sqrt{C(T_1 + T_2 - 2\sqrt{T_1 T_2}}$$