Columbia University
Department of Physics
QUALIFYING EXAMINATION

Wednesday, January 16, 2019
2:00PM to 4:00PM
Modern Physics
Section 4. Relativity and Applied Quantum Mechanics

Two hours are permitted for the completion of this section of the examination. Choose 4 problems out of the 5 included in this section. (You will not earn extra credit by doing an additional problem). Apportion your time carefully.

Use separate answer booklet(s) for each question. Clearly mark on the answer booklet(s) which question you are answering (e.g., Section 4 (Relativity and Applied Quantum Mechanics), Question 2, etc.).

Do **NOT** write your name on your answer booklets. Instead, clearly indicate your **Exam Letter Code**.

You may refer to the single handwritten note sheet on $8\frac{1}{2}$" $\times$ 11" paper (double-sided) you have prepared on Modern Physics. The note sheet cannot leave the exam room once the exam has begun. This note sheet must be handed in at the end of today's exam. Please include your Exam Letter Code on your note sheet. No other extraneous papers or books are permitted.

Simple calculators are permitted. However, the use of calculators for storing and/or recovering formulae or constants is NOT permitted.

Questions should be directed to the proctor.

Good Luck!

1. A cosmic ship with rest-mass m is moving with speed $v = 0.6c$ with respect to an inertial "lab" frame. No external forces act on the ship. At time $t = 0$ the ship switches on a laser with power P_0 and emission frequency ν_0 (both measured in the rest frame of the ship). The laser beam propagates ahead of the ship, in the direction of its motion.

   (a) What is the laser emission frequency ν measured in the lab frame at $t = 0$?

   (b) What is the laser emission power P measured in the lab frame at $t = 0$?

   (c) The laser continues to radiate power $P_0 =$ const., and recoil from the emitted photons decelerates the ship. Find the mass of the ship when it stops in the lab frame.

   (d) Find the proper time τ (measured by the clock on the ship) after which the ship stops in the lab frame.

2. A space explorer leaves Earth in a ship and accelerates at 3 m/s^2 for one year. She then coasts for one year, and spends a third year decelerating at the same rate as the original acceleration ("accerlerates for one year", "coasts for one year" and "spend a third year" all refer to "one year" in the explorer's reference frame). How far from Earth did the explorer get? (Take 1 year $= 3.2 \times 10^7$ s, $c = 3 \times 10^8$ m/s.)

   Here, constant acceleration at some rate a (e.g. 3 m/s^2) means this:

$$-\left(\frac{d^2t}{d\tau^2}\right)^2 + \frac{1}{c^2}\left(\frac{d^2x}{d\tau^2}\right)^2 = \frac{a^2}{c^2},$$

   where t and x are the time and distance according to the Earth frame, and τ is the proper time of the space traveler i.e.

$$-\left(\frac{dt}{d\tau}\right)^2 + \frac{1}{c^2}\left(\frac{dx}{d\tau}\right)^2 = -1.$$

3. A proton of energy E collides with a neutron at rest.

   (a) Calculate the threshold energy E for neutral pion production: $p + n \rightarrow p + n + \pi^0$. You should take the masses of the proton and neutron to be equal. Express your answer in terms of the masses of the particles.

   (b) The neutral pion then decays into two photons: $\pi^0 \rightarrow \gamma + \gamma$. Calculate the energy of a photon produced in such a decay. Express your answer in terms of the masses of the particles.

   You may assume that the photons are emitted parallel and anti-parallel, respectively, to the momentum of the pion in the lab frame. Explain why this will maximize the energy of the parallel photon.

4. One electron is moving in the weak periodic potential of a linear chain along the x axis, with sites located at positions given by

$$x_n = na,$$

where $n = 0, 1, 2, ..., N = L/a \to \infty$, where L is the total length of the chain. The chain has a weak periodic potential, which can be expanded in a Fourier series.

$$V(x) = \sum_m V_m e^{iK_m x} = V(x+a),$$

where $K_m = 2\pi m/a$ and $m = 1, 2, 3, ...$

The wavefunction of the electron states can be expanded as a set of plane waves

$$\psi(x) = \sum_q C(q) e^{iqx},$$

where the expansion coefficients $C(q)$ can be evaluated from the Schroedinger equation using the potential $V(x)$. Hereafter just assume the $C(q)$ are given.

(a) We are interested in the symmetry of the wavefunctions. Writing

$$q = K_m + k,$$

prove that the wave function for states with wave vector k can be written as

$$\psi_k(x) = e^{ikx} u_k(x), \tag{1}$$

i.e. find an expression for $u_k(x)$.

(b) Show that

$$u_k(x) = u_k(x+a). \tag{2}$$

(c) Show that the wave functions given by equations (1) and (2) satisfy

$$\psi_k(x) = \psi_{k+K_m}(x) \tag{3}$$

(d) Equation (3) implies that the energy eigenvalues satisfy:

$$E(k) = E(k+K_m) \tag{4}$$

so that the energy states form a set of bands linked to the index m, i.e., for each m there is an interval of length $\Delta k = 2\pi m/a$ of k values that give distinct energies.

For the lowest band ($m = 0$)

$$E(k) \simeq \frac{\hbar^2 k^2}{2m_0}, \tag{5}$$

where m_0 is the free electron mass.

An electric field is applied. In the so-called semi-classical approximation, the equation for the motion of the electron is given by

$$\hbar \frac{dk}{dt} = -eE \tag{6}$$

Assume that the electron moves only within the next lowest band ($m = 1$). As the electron moves through the states of the band, the electron energy linked to changes in the wave vector k changes periodically as a function of time.

Calculate the frequency of the periodic variation in the electron energy.

5. A beam of electrons with random spins is passed along the y-axis through a Stern-Gerlach apparatus blocking $s_z = -\frac{1}{2}$ states $|\downarrow\rangle$, allowing only $s_z = +\frac{1}{2}$ states $|\uparrow\rangle$ to pass through.

   (a) **What fraction of the initial beam gets through this spin filter?**
   Let us call this filter A.

   (b) A second Stern-Gerlach apparatus—let us call it filter B—is placed in sequence after filter A, but rotated by 180 degrees around the beam axis so only $s_z = -\frac{1}{2}$ states $|\downarrow\rangle$ are allowed through.

   What fraction of the initial beam gets through both filters A and B?

   (c) Another Stern-Gerlach filter (C) is placed *between* filter A and B, now rotated by 90 degrees around the beam axis so only $s_x = +\frac{1}{2}$ state $|\rightarrow\rangle$ are allowed through.

   What fraction of the initial beam gets through the full sequence of three filters A → C → B?

   (d) More generally, say we have a sequence of two subsequent filters rotated by an angle ϕ along the beam axis with respect to each other.

   What fraction of the beam that survived the first filter also survives the second filter?

   (e) Finally, consider a setup in which N filters are placed between A and B, each rotated with respect to the previous one by an angle $\phi = \dfrac{\pi}{N+1}$.

   What fraction of the beam passes through the full sequence of filters in the limit $N \to \infty$?

Question 1 Solution

(a) Each photon emitted by the laser has energy $E_0 = h\nu_0$ and momentum $p_0 = E_0/c$ in the ship frame. The photon 4-momentum is $(p_0, p_0, 0, 0)$, where the x-axis is chosen along the laser beam direction. The ship is moving along the x-axis, and Lorentz transformation of the photon four-momentum gives the photon energy in the lab frame:

$$E = \gamma E_0 + \gamma v p_0 \implies \nu = \gamma(1+\beta)\nu_0 = 2\nu_0,$$

where $\beta = v/c = 3/5$, and $\gamma = (1-\beta^2)^{-1/2} = 5/4$.

(b) The laser power in the ship frame may be written as $P_0 = E_0/\delta\tau$, where $\delta\tau$ is the time interval between the emission of successive photons. The power in the lane frame is $P = E/\delta t$, where $\delta t = \gamma\,\delta\tau$. This gives

$$P = \frac{E}{\delta t} = \frac{\gamma(1+\beta)E_0}{\gamma\,\delta\tau} = (1+\beta)P_0 = \frac{8}{5}P_0,$$

(c) In the final state the ship momentum vanishes in the lab frame, and so its energy is $E_f = m_f c^2$. The initial energy was $E = \gamma mc^2$. The lost energy $\Delta E = E_i - E_f$ is related to the lost momentum $\Delta p = m\gamma v$ by $\Delta E = c\Delta p$, since both ΔE and Δp are carried by photons in the laser beam. This gives

$$m_f = \frac{E_f}{c^2} = \frac{\gamma mc^2 - \Delta E}{c^2} = m\,\gamma(1-\beta) = \frac{m}{2}.$$

(d) The emitted power P_0 implies "burning" the ship rest mass with rate

$$\frac{dm}{d\tau} = -\frac{P_0}{c^2} \implies \tau = \frac{(m-m_f)c^2}{P_0} = \frac{mc^2}{2P_0}.$$

Question 2 Solution

The definition of proper acceleration (the acceleration that the person in the spaceship feels) is $a = c\frac{d\eta}{d\tau}$, where η is the rapidity. Remember that $\tanh\eta = \beta$. Since the proper acceleration is constant during the three intervals, we can integrate both sides to find $\eta = a\tau/c$.

Let's start by evaluating how far the explorer gets with respect to Earth. We compute

$$dx = \beta c\, dt = \tanh\eta\, c\gamma\, d\tau = c\tanh\eta\, \cosh\eta\, d\tau = c\sinh\eta\, d\tau,$$

where dx is in the Earth's frame. So, for the accelerating plus decelerating segments we have

$$\Delta x = 2c\int_0^{\tau=1\text{ year}} \sinh\left(\frac{a}{c}\tau\right)\, d\tau = 2\frac{c^2}{a}\left(\cosh\left(\frac{a}{c}(1\text{ year})\right) - 1\right) = \cdots = 0.3\times 10^{16}\text{ m}.$$

To that, we need to add one year of coasting at maximal velocity. We compute the rapidity for this part of the trip:

$$\eta_{\text{max}} = \int_0^{\tau=1\text{ year}} d\eta = \int_0^{\tau=1\text{ year}} \frac{a}{c}\, d\tau = 0.32,$$

so that

$$\Delta x_{\text{coast}} = \beta_{\text{max}} c\Delta t = \beta_{\text{mas}}\gamma_{\text{max}} c\Delta\tau = c\sinh\eta_{\text{max}}\,\Delta\tau = 3\times 10^{15}\text{ m}.$$

The total distance traveled is about 6×10^{15} m which is about 0.6 light years.

Useful expressions:

$$t = \frac{c}{a}\sinh[a\tau/c] + \text{const.},\quad x = \frac{c^2}{a}\cosh[a\tau/c] + \text{const.},\quad \frac{dx}{dt} = c\,\tanh[a\tau/c]$$

Question 3 Solution

(a) W set $c = 1$. In the lab frame, the four-momenta of the proton and neutron before the collision are, respectively,

$$(E, \mathbf{p}) \qquad (m_n, \mathbf{0}).$$

In the center of mass frame, they are

$$(E_{\text{CM}}, \mathbf{p}_{\text{CM}}) \qquad (E_{\text{CM}}, -\mathbf{p}_{\text{CM}})\,.$$

Note that the energies are equal in the lab frame only if we take $m_p = m_n$. If E is the threshold energy for the production of a pion is, then all three particles must be at rest after the collision in the lab frame. Thus by conservation of energy,

$$2E_{\text{CM}} = m_p + m_n + m_\pi = 2m_p + m_\pi.$$

We also see that

$$E_{\text{CM}}^2 = m_p^2 + |\mathbf{p}_{\text{CM}}|^2.$$

Take the inner product of the proton and neutron momenta before the collision in both frames yields

$$Em_n = E_{\text{CM}}^2 + |\mathbf{p}_{\text{CM}}|^2,$$

which by the previous equation becomes

$$Em_n = 2E_{\text{CM}}^2 - m_p^2.$$

Using the very first equation, we find

$$\begin{aligned} E &= \frac{2E_{\text{CM}}^2}{m_p} - m_p \\ &= \frac{2}{m_p}\left(m_p^2 + m_p m_\pi + \frac{1}{4}m_\pi^2\right) - m_p \\ &= m_p + 2m_\pi + \frac{m_\pi^2}{2m_p} \\ &\approx 1.2\,\text{GeV}. \end{aligned}$$

(b) Taking the pion to be moving along the x axis and the photons to be emitted in the x-y plane, it is clear that the x components of the photons' momenta must add up to the momentum of the pion, while the y components must cancel. We note that if the energy of one of the photons (call it photon 1) is to be maximized, it must also have a large momentum. This can be done by letting the x component of the momentum of photon 2 be negative, which allows us to make the x component of photon 1 large. However, the magnitudes of the two momenta must add up to the energy of the pion, so we cannot make the momentum of photon 1 arbitrarily large. It then follows that we should make the y components of the momenta as small as possible, since they contribute to the total energy but not the total momentum. Thus we see that the energy of photon 1 is maximized when it is emitted parallel to the pion's momentum and photon 2 is emitted in the opposite direction.

Considering the maximal case described above, the four-momenta of the pion and the two photons in the lab frame are

$$(E_\pi, p_\pi, 0, 0) \qquad (E_{\gamma 1}, E_{\gamma 1}, 0, 0) \qquad (E_{\gamma 2}, -E_{\gamma 2}, 0, 0)$$

By conservation of four-momentum,

$$E_{\gamma 1} + E_{\gamma 2} = E_\pi, \qquad E_{\gamma 1} - E_{\gamma 2} = p_\pi.$$

This yields

$$E_{\gamma 1} = \frac{1}{2}\left(E_\pi + p_\pi\right) = \frac{1}{2}\left(E_\pi + \sqrt{E_\pi^2 - m_\pi^2}\right).$$

Since $E_\pi = \gamma m_\pi$, this becomes

$$E_{\gamma 1} = \frac{1}{2} m_\pi \left(\gamma + \sqrt{\gamma^2 - 1}\right) = \frac{1}{2} m_\pi \gamma \left(1 + \beta\right).$$

Here γ is the Lorentz factor for the relative velocity of the lab frame and the center of mass frame for part (b). However, this is the same Lorentz factor for the lab frame and the center of mass frame for part (a), since the pion is at rest in both center of mass frames. Thus if we consider the neutron in the center of mass frame of part (a), we can write

$$E_{\text{CM}} = \gamma m_n = \gamma m_p.$$

Since

$$E_{\text{CM}} = m_p + \frac{1}{2} m_\pi,$$

we find

$$\gamma = 1 + \frac{m_\pi}{2 m_p}.$$

This implies

$$\beta = \frac{\sqrt{\gamma^2 - 1}}{\gamma} = \frac{\sqrt{m_\pi \left(4 m_p + m_\pi\right)}}{2 m_p + m_\pi}.$$

Thus

$$E_{\gamma 1} = \frac{1}{2} m_\pi \left(1 + \frac{m_\pi}{2 m_p}\right)\left(1 + \frac{\sqrt{m_\pi \left(4 m_p + m_\pi\right)}}{2 m_p + m_\pi}\right).$$

Since $m_\pi \ll m_p$, this is approximated by

$$E_{\gamma 1} \approx \frac{1}{2} m_\pi \left(1 + \frac{m_\pi}{2 m_p}\right)\left(1 + \sqrt{\frac{m_\pi}{m_p}}\right) \approx 100\,\text{MeV}.$$

Question 4 Solution

(a) We write

$$\psi(x) = \sum_k \sum_m C(K_m + k) e^{i(k+K_m)x} = \sum_k \psi_k(x),$$

where

$$\psi_k(x) = e^{ikx} \sum_m C(K_m + k) e^{iK_m x} = e^{ikx} u_k(x) \tag{1}$$

where

$$u_k(x) = \sum_m C(K_m + k) e^{iK_m x}$$

(b) We compute

$$u_k(x+a) = \sum_m C(K_m + k) e^{iK_m x} e^{iK_m a} = \sum_m C(K_m + k) e^{iK_m x} = u_k(x),$$

where in the second to last equality we have used the fact that $e^{iK_m a} = 1$.

(c) Using equation (1) with $k \to k + K_m$,

$$\begin{aligned}
\psi_{k+K_m} &= e^{ikx} e^{iK_m x} \sum_{m'} C(K_{m'} + K_m + k) e^{iK_{m'} x} \\
&= e^{ikx} \sum_{m'} C(K_{m'} + K_m + k) e^{i(K_{m'} + K_m)x} \\
&= e^{ikx} \sum_{m'} C(K_{m'+m} + k) e^{iK_{m'+m} x} \\
&= e^{ikx} \sum_{m} C(K_m + k) e^{iK_m x} \qquad \text{(relabeling } m' + m \text{ with } m\text{)} \\
&= \psi_k(x).
\end{aligned}$$

(d) If T is the period of the energy oscillation, then the period in k-space is

$$\Delta k = \frac{2\pi}{a}$$

From the equation given in the problem

$$\hbar \frac{dk}{dt} = -eE$$

replacing $dk/dt = \Delta k / T$, we obtain

$$\hbar \Delta k = -eET \implies T = \left| \frac{\hbar \Delta k}{eE} \right| = \left| -\frac{2\pi\hbar}{eaE} \right|$$

Thus the frequency is

$$\omega = \frac{2\pi}{T} = \left| \frac{-eaE}{\hbar} \right|.$$

Question 5 Solution

(a) $\boxed{1/2}$ of the beam passes through.

(b) All electrons making it through A are in the state $|\uparrow\rangle = \binom{1}{0}$. The probability of measuring spin down is then $|\langle\downarrow|\uparrow\rangle|^2 = 0$, so $\boxed{\text{nothing}}$ passes through both A and B.

(c) All electrons making it through A are in the state $|\uparrow\rangle = \binom{1}{0}$. The relevant spin eigenstates for filter C are eigenstates of $S_x = \frac{1}{2}\sigma_x = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, that is $|\rightarrow\rangle = \frac{1}{\sqrt{2}}\binom{1}{1}$ with eigenvalue $s_x = +\frac{1}{2}$, and $|\leftarrow\rangle = \frac{1}{\sqrt{2}}\binom{1}{-1}$ with eigenvalue $s_x = -\frac{1}{2}$. The probability to find $s_x = +\frac{1}{2}$ for an electron in the $|\uparrow\rangle$ state is therefore

$$|\langle\rightarrow|\uparrow\rangle|^2 = \frac{1}{2}\begin{pmatrix} 1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{2}.$$

Thus $1/2 \times 1/2 = 1/4$ of the initial beam passes through both A and C. Because the final filter B in the sequence is rotated by the same amount with respect to C as C is rotated with respect to A, filter B will again cut the beam by a factor 1/2, so $\boxed{1/8}$ of the original beam passes through all three filters.

(d) Without loss of generality we can assume the spin state on which the first filter projects to be $|\uparrow\rangle = \binom{1}{0}$. The spin state $|\phi\rangle$ on which the second filter projects is then obtained from $|\uparrow\rangle$ by a rotation by an angle ϕ around the y-axis:

$$|\phi\rangle = \exp\left[-i\phi\tfrac{1}{2}\sigma_y\right]\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \exp\left[-\frac{\phi}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right]\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \cos\frac{\phi}{2} \\ \sin\frac{\phi}{2} \end{pmatrix}. \tag{2}$$

The probability of surviving the second filter for an electron that survived the first filter is therefore $P(\phi) = |\langle\phi|\uparrow\rangle|^2 = \boxed{\cos^2\frac{\phi}{2}}$.

(e) As before, half of the beam survives A. A fraction $\cos^2\frac{\pi}{2(N+1)}$ survives each subsequent filter, so in total a fraction $\frac{1}{2} \times \left(\cos^2\frac{\pi}{2(N+1)}\right)^{N+1}$ survives. In the limit $N \to \infty$, we have

$$\left(\cos^2\frac{\pi}{2(N+1)}\right)^{N+1} \approx \left(1 + \frac{\pi^2}{8(N+1)^2}\right)^{N+1} \approx e^{\frac{\pi^2}{8(N+1)}} \to 1.$$

Therefore in this limit the subsequent filters allow the full beam to pass, and altogether a fraction $\boxed{1/2}$ of the initial beam survives.