Columbia University
Department of Physics
QUALIFYING EXAMINATION

Monday, January 14, 2019
2:00PM to 4:00PM
Classical Physics
Section 2. Electricity, Magnetism & Electrodynamics

Two hours are permitted for the completion of this section of the examination. Choose 4 problems out of the 5 included in this section. (You will not earn extra credit by doing an additional problem). Apportion your time carefully.

Use separate answer booklet(s) for each question. Clearly mark on the answer booklet(s) which question you are answering (e.g., Section 2 (Electricity etc.), Question 2, etc.).

Do **NOT** write your name on your answer booklets. Instead, clearly indicate your **Exam Letter Code**.

You may refer to the single handwritten note sheet on $8\frac{1}{2}$" $\times$ 11" paper (double-sided) you have prepared on Classical Physics. The note sheet cannot leave the exam room once the exam has begun. This note sheet must be handed in at the end of today's exam. Please include your Exam Letter Code on your note sheet. No other extraneous papers or books are permitted.

Simple calculators are permitted. However, the use of calculators for storing and/or recovering formulae or constants is NOT permitted.

Questions should be directed to the proctor.

Good Luck!

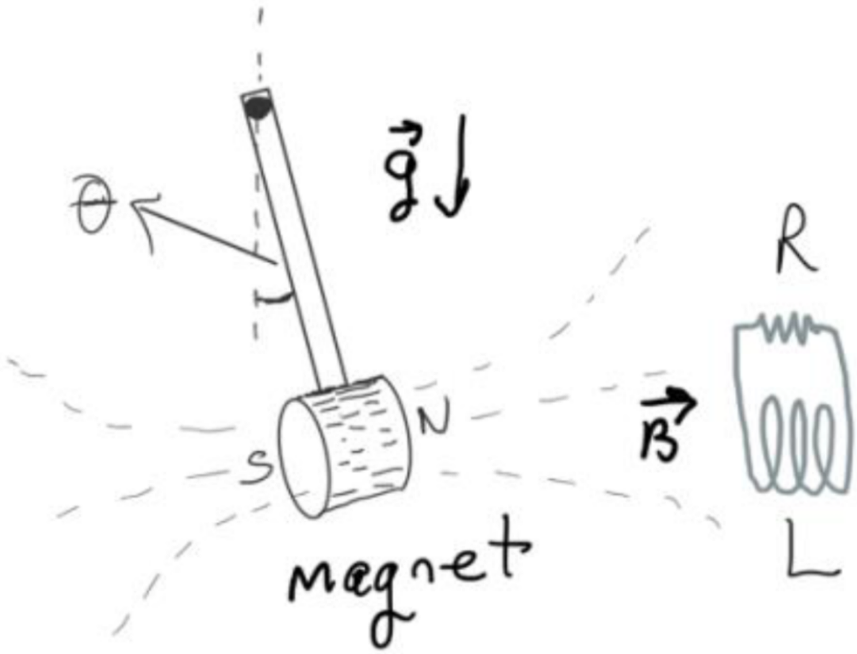
1. Two polarizable atoms, A and B, are separated by a fixed distance z along the z-axis. Atom A is polarized with a dipole moment $\mathbf{p}_A = p_A\hat{\mathbf{z}}$ as a result of the electric field due to the dipole moment $\mathbf{p}_B = p_B\hat{\mathbf{z}}$ of the atom B. Likewise, the dipole moment $\mathbf{p}_B$ is the result of polarization by the electric field due to $\mathbf{p}_A$. No other charges or electric fields are present in the problem other than the two dipoles.

   (a) Are $\mathbf{p}_A$ and $\mathbf{p}_B$ parallel or anti-parallel to one another? Explain.

   (b) Calculate the force between the two electric dipoles in terms of the givens (p_A, p_B, z).

   (c) Is the force calculated in part (b) attractive or repulsive?

2. A magnet of mass m is attached to a rigid rod of length ℓ to create a pendulum as shown in the figure below. The vertical gravitational acceleration is given by g.



A coil of inductance L is brought to the vicinity of the pendulum. The magnetic flux ϕ through the coil changes as the pendulum swings back and forth. ϕ is related to the instantaneous angle of the pendulum $\theta(t)$ by

$$\phi(t) = \phi_a + \phi_b \theta(t).$$

A resistor R is attached to the terminals of the inductance coil so that current flows through the coil in response to the changing magnetic flux.

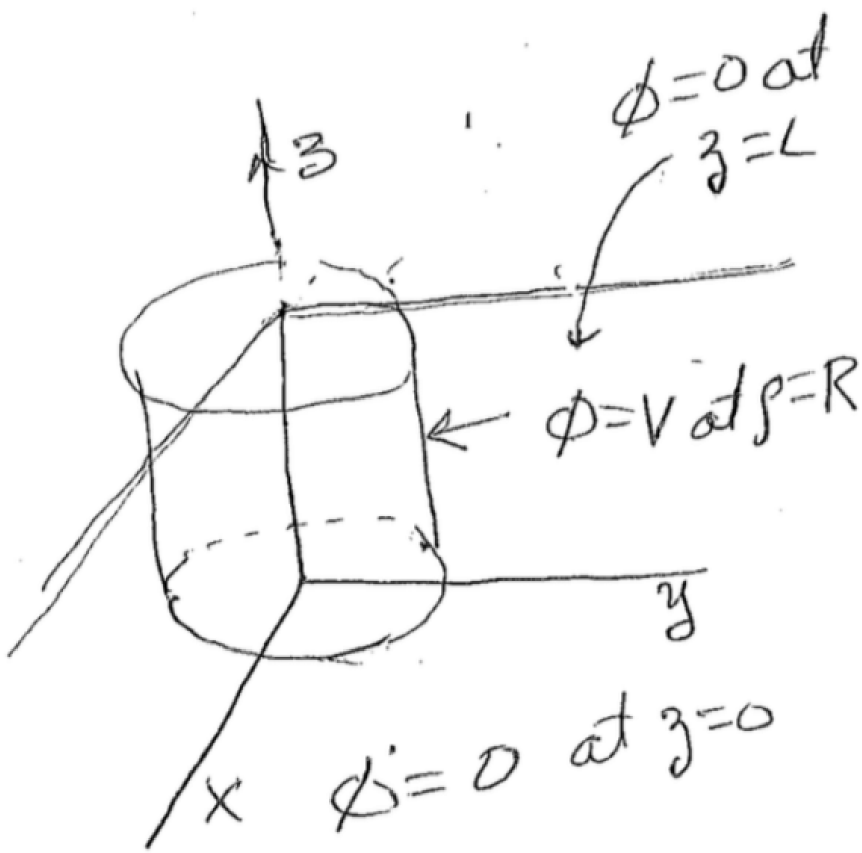
Determine an approximate expression for the amplitude $\Theta(t)$ of the oscillation angle θ in terms of ℓ, R, m, ϕ_b, and the initial amplitude Θ_0.

You can assume $\theta(t)$ has the form $\theta(t) = \Theta(t)\cos(\omega t)$ with Θ small. You can assume that the magnet and coil are weakly couped such that $\Theta(t)$ does not change significantly during a single oscillation cycle, i.e., $\dot{\Theta} << \Theta\omega$, and that the oscillation frequency of the pendulum is not changed significantly by the presence of the coil.

3. Consider a cylinder of radius $\rho = R$ and height $z = L$ as shown in the figure below. The electric potential ϕ on the outer surface of the cylinder is a specified function of height:

$$V(z) = \phi\,(\rho = R, 0 < z < L)\,.$$

The top and bottom of the cylinder are bounded by infinite sheets (the surfaces $z = 0$ and $z = L$, respectively) held at zero electric potential $\phi = 0$.



Find a solution for the electric potential $\phi(\rho, z)$ in the region $(\rho > R, 0 < z < L)$, i.e., outside the cylinder and between the plates. Assume that no charge resides in this region. You may express your result in terms of special functions.

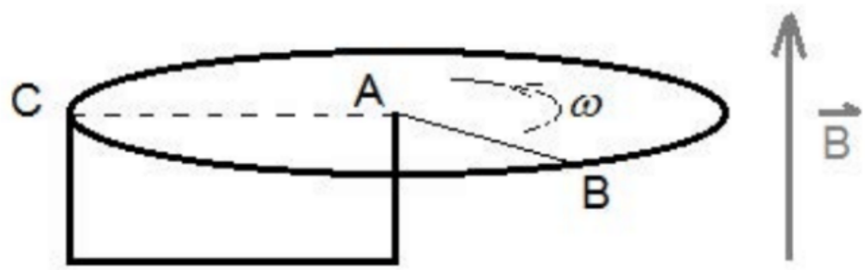
Hints: The Laplacian of a scalar function f is expressed in cylindrical coordinates as

$$\nabla^2 f = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial f}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 f}{\partial \varphi^2} + \frac{\partial^2 f}{\partial z^2}.$$

The modified Bessel functions $K_n(x)$ satisfy this differential equation:

$$\frac{d^2 K_n(x)}{dx^2} + \frac{1}{x}\frac{dK_n(x)}{dx} - \left(1 + \frac{n^2}{x^2}\right) K_n(x) = 0.$$

4. A horizontal ring of radius r is placed in a uniform vertical magnetic field $\mathbf{B}$. A bar of resistance R connects the center of the ring (point A in the figure) to a point B on the ring. The bar rotates about A, moving with constant angular velocity ω. The angular velocity of the point B is parallel to the magnetic field $\mathbf{B}$. Point A is also connected to the ring by a stationary wire at the point C. Both the ring and the wire joining points A and C have negligible resistance. The moving bar makes frictionless electrical contact at each of its ends.

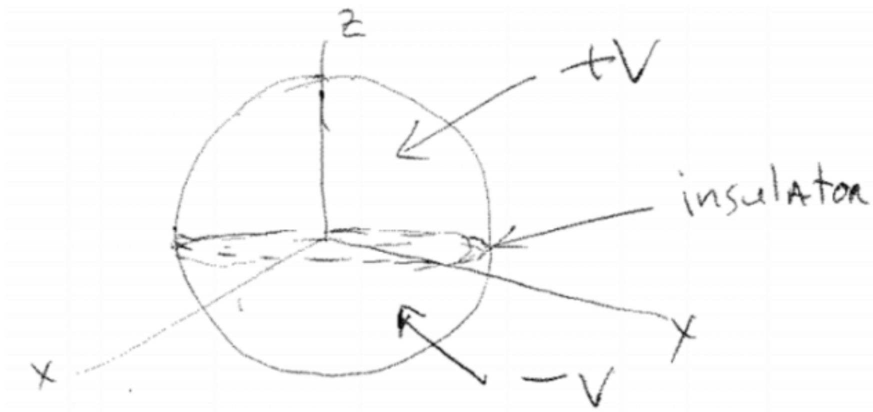


(a) Find the current flowing in the bar joining the points A and B.

(b) Find the rate of energy loss in the moving bar due to Ohmic heating.

(c) Compare your answer to part (b) with the work needed to rotate the wire.

5. Consider a hollow conducting sphere of radius a made up of two hemispherical sections separated by a thin insulating ring. The upper hemisphere is held at potential $+V$ and the lower hemisphere at potential $-V$.



(a) Find expressions for the interior potential ($r < a$) and the exterior potential ($r > a$) in an appropriate series expansion. Your final answer may include unevaluated definite integrals over special functions.

(b) Explicitly evaluate the two lowest order terms for the interior and exterior potential.

You may find the following properties of the Legendre polynomials useful:

$$P_n(-x) = (-1)^n P_n(x)$$

$$\int_{-1}^{1} P_m^2(x)\, dx = \frac{2}{2m+1}.$$

Question 1 Solution

(a) $\mathbf{p}_A$ and $\mathbf{p}_B$ are parallel to each other.

For instance, consider that if $\mathbf{p}_{\text{A}}$ points in the $+\hat{\mathbf{z}}$ direction, then its electric field along the z-axis also points in the $+\hat{\mathbf{z}}$. Since induced electric dipoles align with the external field responsible for inducing them, then the electric dipole of $\mathbf{p}_B$ will also point in the $+\hat{\mathbf{z}}$. direction, i.e. parallel to $\mathbf{p}_A$. Likewise, the electric field due to $\mathbf{p}_B$ also points along its direction $+\hat{\mathbf{z}}$, such that $\mathbf{p}_A$ is also induced to point parallel to $+\hat{\mathbf{z}}$, consistent with the original assumption.

(b) Consider that atom A is located at the origin and atom B is located at $z\hat{\mathbf{z}}$. The electric field of the (pure) dipole moment $\mathbf{p}_A = p_A\hat{\mathbf{z}}$ along the z-axis is given by

$$\mathbf{E}_{\text{A}} = \frac{p_A}{2\pi\epsilon_0 z^3}\hat{\mathbf{z}}$$

The force on atom B is then given by

$$\mathbf{F}_{\text{B}} = (\mathbf{p}_B \cdot \nabla)\mathbf{E}_{\text{A}} = p_B\frac{\partial \mathbf{E}_A}{\partial z}\hat{\mathbf{z}} = -\frac{3p_A p_B}{2\pi\epsilon_0 z^4}\hat{\mathbf{z}}.$$

(c) The force is **attractive** and is known as the **Van Der Waals** force. The result from part (b) shows that force on atom B points in the negative $\hat{\mathbf{z}}$ direction towards atom A. Likewise, one could show that the force on atom A is in the $+\hat{\mathbf{z}}$ direction.

solenoid.jpg in tex file

Question 2 Solution

The electromotive force at the inductor is

$$V = -\frac{d\phi}{dt} = -\phi_b \frac{d\theta}{dt} \tag{1}$$

We write the harmonic dependence of the oscillation angle as

$$\theta(t) = \Theta(t)\cos(\omega t), \tag{2}$$

where $\omega = \sqrt{g/l}$ is the oscillation frequency of the pendulum.
Substituting equation (2) into equation (1) we obtain

$$V = \phi_b \omega \Theta(t)\sin(\omega t),$$

where we have neglected the $\dot{\Theta}$ because it evolves on a timescale much longer than an oscillation period.

The average power dissipated at the resistor is given by

$$P = \frac{V^2}{2R} = \frac{\Theta^2 \phi_b^2 \omega^2}{2R} \tag{3}$$

Due to energy conservation the dissipated power will lead to a decay in the oscillation amplitude of the pendulum. To determine the decay rate, first express the energy stored in the pendulum

$$U = mgl\,(1 - \cos(\Theta)) \approx \frac{1}{2} mgl\Theta^2.$$

From energy conservation, the power dissipated in the resistor must equal that leaving the pendulum,

$$P = -\frac{dU}{dt} = -mgl^2\Theta\frac{d\Theta}{dt}$$

or using equation (3) for P,

$$\frac{\Theta^2\phi_b^2\omega^2}{2R} = -mgl\Theta\frac{d\Theta}{dt}$$

$$\frac{d\Theta}{dt} = -\frac{\phi_b^2\omega^2}{2mglR}\Theta.$$

Hence the amplitude is an exponentially decaying function of time.

$$\Theta(t) = \Theta_0 \exp\left[-\frac{\phi_b^2\omega^2}{2mglR}t\right].$$

Substituting $\omega = \sqrt{g/l}$ we find

$$\Theta(t) = \Theta_0 \exp\left[-\frac{\phi_b^2}{2ml^2R}t\right],$$

which notably is independent of g.

Question 3 Solution

We are looking for axisymmetric ($\partial/\partial\phi = 0$) solutions to the LaPlace equation in cylindrical coordinates

$$\nabla^2\phi = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\Phi}{\partial\rho}\right) + \frac{\partial^2\Phi}{\partial z^2} = 0 \tag{1}$$

Trying separable solutions of the form

$$\phi(\rho, z) = \mathcal{Z}(z)\mathcal{R}(\rho),$$

we obtain a SHO equation for $\mathcal{Z}$,

$$\frac{d^2\mathcal{Z}}{dz^2} = -\mathcal{C}\mathcal{Z},$$

where $\mathcal{C}$ is a constant. The normalized solution of this which match the boundary conditions $\phi(z = 0) = \phi(z = L) = 0$ are sines,

$$\mathcal{Z} = \sqrt{\frac{2}{L}}\sin\left(\frac{m\pi z}{L}\right),$$

where m is an integer. Therefore, we can write the general solution to equation (1) as

$$\phi(\rho, z) = \sum_{m=1}^{\infty}\mathcal{R}_m(\rho)\sqrt{\frac{2}{L}}\sin\left(\frac{m\pi z}{L}\right),$$

where the $\mathcal{R}_m$ obey the differential equation

$$\left[\frac{d^2}{d\rho^2} + \frac{1}{\rho}\frac{d}{d\rho} - \left(\frac{m\pi}{L}\right)^2\right]\mathcal{R}_m(\rho) = 0$$

Defining the dimensionless variable $J \equiv m\pi\rho/L$, this can be expressed as

$$\left[\frac{d^2}{dJ^2} + \frac{1}{J}\frac{d}{dJ} - 1\right]\mathcal{R}_m(J) = 0$$

This is the Bessel equation of order zero. We can write the solutions K_0 as

$$\mathcal{R}_m(\rho) = C_m K_0(J) = C_m K_0\left(\frac{m\pi\rho}{L}\right),$$

where C_m are constants. Thus

$$\phi = \sum_{m=1}^{\infty} C_m\sqrt{\frac{2}{L}}\sin\left(\frac{m\pi z}{L}\right)K_0\left(\frac{m\pi\rho}{L}\right).$$

To figure out the C_m we need to apply the final boundary condition on the surface of the cylinder,

$$V(z) = \phi(\rho = R, z) = \sum_{m=1}^{\infty} C_m\sqrt{\frac{2}{L}}\sin\left(\frac{m\pi z}{L}\right)K_0\left(\frac{m\pi R}{L}\right).$$

Multiplying both sides of this equation by $\sin\left(\dfrac{m'\pi z}{L}\right)$ and integrating over z,

$$\int_0^L V(z)\sin(m'\pi z/L)dz = \sum_{m=1}^{\infty}\sqrt{\frac{2}{L}}C_m K_0\left(\frac{m\pi R}{L}\right)\int_0^L \sin(m'\pi z/L)\sin(m\pi z/L)dz,$$

and using the orthogonality properties of the sines, i.e.

$$\int_0^L \sin\left(\frac{m\pi z}{L}\right)\sin\left(\frac{m'\pi z}{L}\right)\, dz = \frac{L}{2}\delta_{m,m'}$$

we then obtain

$$C_{m'} = \frac{\displaystyle\int_0^L \sqrt{\frac{2}{L}}\sin\left(\frac{m'\pi z}{L}\right)V(z)\,dz}{K_0(m\pi R/L)}.$$

Putting everything together,

$$\begin{aligned}\phi &= \sum_{m=1}^{\infty}\frac{K_0(m\pi\rho/L)}{K_0(m\pi R/L)}\left(\int_0^L \sqrt{\frac{2}{L}}\sin\left(\frac{m\pi z'}{L}\right)V(z')\,dz'\right)\sqrt{\frac{2}{L}}\sin\left(\frac{m\pi z}{L}\right)\\ &= \frac{2}{L}\sum_{m=1}^{\infty}\frac{K_0(m\pi\rho/L)}{K_0(m\pi R/L)}\left(\int_0^L \sin\left(\frac{m\pi z'}{L}\right)V(z')\,dz'\right)\sin\left(\frac{m\pi z}{L}\right).\end{aligned}$$

Question 4 Solution

(a) Identify two areas bounding the segment of the ring joining C and B and the two radii AB and AC. One area is increasing at the rate $\omega r^2/2$ and the other is decreasing at the same rate. The magnetic flux through either closed circuit is changing. Applying Faraday's law,

$$\mathcal{E} = -\frac{1}{c}\frac{d\Phi_B}{dt},$$

to either area determines the current I flowing in the bar:

$$I = \frac{|\mathcal{E}|}{R} = \frac{1}{cR}\frac{d}{dt}\Phi_B = \frac{\omega r^2 B}{2cR}.$$

(b) The power dissipated in the bar is then

$$I^2 R = \frac{\omega^2 r^4 B^2}{4c^2 R}.$$

(c) The power consumed rotating the bar is $P = \tau\omega$, where τ is the torque given by

$$\tau = \int_0^r \frac{df}{dr} r'\,dr' = \int_0^r \frac{IB}{c} r'\,dr' = \frac{IB}{2c} r^2,$$

where $df/dr = IB/c$ is the Lorentz force per unit length. Thus,

$$\tau\omega = \frac{IB}{2c} r^2 \omega = \frac{\omega^2 r^4 B^2}{4c^2 R},$$

equal to the rate of Ohmic energy loss given in part (b).

Question 5 Solution

(a) We are looking for solutions to the axisymmetric ($\partial/\partial\phi = 0$) LaPlace equation

$$\nabla^2\phi = 0$$

in spherical coordinates (r, θ, ϕ).

Separation of variables $\phi(r, \theta) = \mathcal{R}(r)\Theta(\theta)$ gives two solutions for the radial function,

$$\mathcal{R}(r) = r^{-(m+1)}, r^m$$

and convergent solutions for the angular dependence,

$$\Theta(\theta) = P_m(\theta),$$

where P_m are the Legendre polynomials of integer order $m \geq 0$.

Since r^m blows up as $r \to \infty$, the exterior ($r > a$) solutions must be of the form

$$\phi_e(r, \theta) = \sum_{m=0}^{\infty} A_m r^{-(m+1)} P_m(\cos\theta), \tag{1}$$

where A_m are constants.

On the other hand, since $r^{-(m+1)}$ blows up at $r = 0$, the interior ($r < a$) solutions must be of the form

$$\phi_i(r, \theta) = \sum_{m=0}^{\infty} B_m r^m P_m(\cos\theta), \tag{2}$$

where the B_m are constants.

The boundary conditions in both interior and exterior cases are

$$\phi_i(r = a, \theta) = \phi_e(r = a, \theta) = +V, \quad 0 < \theta < \pi/2$$

$$\phi_i(r = a, \theta) = \phi_e(r = a, \theta) = -V, \quad \pi/2 < \theta < \pi$$

Consider the exterior potential first. Multiplying equation (1) by $P_n(x)$, where in general n is a different integer from m, and evaluating at $r = a$ and integrating over $\theta = 0, \pi$ (equivalently, $x = \cos\theta$ from -1 to 1) gives

$$\int_0^1 +V P_n(x)\,dx + \int_{-1}^0 (-V)P_n(x)\,dx = \sum_{m=0}^{\infty} \int_{-1}^1 A_m a^{-(n+1)} P_m(x)P_n(x)\,dx \tag{3}$$

Define an integral of the Legendre polynomial as,

$$\Phi_n \equiv \int_0^1 P_n(x)\,dx$$

In terms of this definition, equation (3) then becomes

$$\begin{aligned} V[\Phi_n - (-1)^n \Phi_n] &= \sum_{m=0}^{\infty} \frac{2A_m}{(2m+1)} \delta_{mn} a^{-(m+1)} \\ &= \frac{2A_n}{(2n+1)} a^{-(n+1)} \end{aligned}$$

where we have used the orthogonality properties of the Legendre polynomials

$$\int_{-1}^{1} P_m P_n \, dx = \frac{2}{2m+1} \delta_{mn}$$

and the fact that

$$\begin{aligned} \int_{-1}^{0} P_n(x) \, dx &= -\int_{1}^{0} P_n(-x) \, dx \\ &= \int_{0}^{1} P_n(-x) \, dx \\ &= (-1)^n \int_{0}^{1} P_n(x) \, dx \\ &= (-1)^n \Phi_n, \end{aligned}$$

The third step makes use of the symmetry property of the LeGendre polynomials that $P_n(-x) = (-1)^n P_n$.

We note that

$$\Phi_n - (-1)^n \Phi_n = \begin{cases} 2\Phi_n, & n \text{ odd} \\ 0 & n \text{ even.} \end{cases}$$

Solving for A_n yields

$$A_n = \begin{cases} (2n+1) a^{n+1} V \Phi_n, & n \text{ odd} \\ 0, & n \text{ even} \end{cases}$$

Substituting back into equation (1) we have

$$\phi_e(r, \theta) = V \sum_{m \text{ odd}} (2m+1) \left(\frac{a}{r}\right)^{m+1} \Phi_m P_m(\cos\theta), \tag{4}$$

For the interior potential ϕ_i (eq. 1) you can see by inspection that we must have

$$B_m = \begin{cases} (2m+1) a^{-m} V \Phi_m, & m \text{ odd} \\ 0, & m \text{ even} \end{cases}$$

Thus the interior potential is

$$\phi_i(r, \theta) = \sum_{m \text{ odd}} (2m+1) \left(\frac{r}{a}\right)^{m} V \Phi_m P_m(\cos\theta).$$

(b) The first two odd Legendre polynomials are given by

$$P_1 = x, P_3 = (1/2)(5x^3 - 3x),$$

such than

$$\Phi_1 = \int_0^1 P_1\,dx = \frac{1}{2}, \quad \Phi_3 = \int_0^1 P_3\,dx = -\frac{1}{8}.$$

Therefore, we can expand (4) for the exterior potential to leading order in a/r

$$\phi_e(r,\theta) = V\left[\frac{3}{2}\left(\frac{a}{r}\right)^2 P_1(\cos\theta) - \frac{7}{8}\left(\frac{a}{r}\right)^4 P_3(\cos\theta) + ...\right]$$

Likewise, for the interior potential we have

$$\phi_i = V\left[\frac{3}{2}\left(\frac{r}{a}\right) P_1(\cos\theta) - \frac{7}{8}\left(\frac{r}{a}\right)^3 P_3(\cos\theta) + ...\right]$$