# Columbia University
# Department of Physics
# QUALIFYING EXAMINATION

Monday, January 14, 2019
10:00AM to 12:00PM
Classical Physics
Section 1. Classical Mechanics

Two hours are permitted for the completion of this section of the examination. Choose 4 problems out of the 5 included in this section. (You will not earn extra credit by doing an additional problem). Apportion your time carefully.

Use separate answer booklet(s) for each question. Clearly mark on the answer booklet(s) which question you are answering (e.g., Section 1 (Classical Mechanics), Question 2, etc.).

Do **NOT** write your name on your answer booklets. Instead, clearly indicate your **Exam Letter Code**.

You may refer to the single handwritten note sheet on $8\frac{1}{2}$" $\times$ 11" paper (double-sided) you have prepared on Classical Physics. The note sheet cannot leave the exam room once the exam has begun. This note sheet must be handed in at the end of today's exam. Please include your Exam Letter Code on your note sheet. No other extraneous papers or books are permitted.

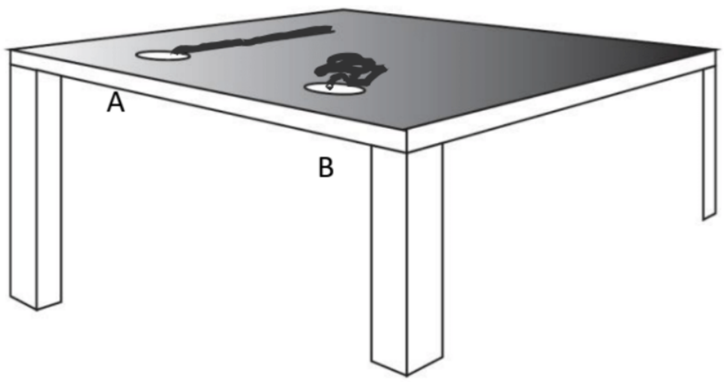
Simple calculators are permitted. However, the use of calculators for storing and/or recovering formulae or constants is NOT permitted.

Questions should be directed to the proctor.

Good Luck!

1. Consider two uniform ropes (A and B), each of length $L$, and each of which can pass through holes in a table. A small piece of each rope extends through each of the holes. Each rope is released from rest and starts to fall through its hole in the table. In the case of rope A, the rope starts stretched out along the table and moves as a whole towards the hole. In the case of rope B, the rope starts in a heap near the hole. As rope B unwinds from the heap, the part on the table stays at rest and only the part that has passed through the hole moves downward. Friction does not act on either rope.

   Consider the speeds of rope A and rope B at the instant the ropes lose contact with the table. Are the speeds equal, or is the speed of rope A larger than that of rope B, or vice versa? Prove your answer.



2. A mass $m$ moves in a circular orbit of radius $r_0$ under the influence of a central force whose potential is given by $V(r) = -\frac{km}{r^n}$, where $k$ is a positive constant and $n$ is a positive integer.

   (a) Determine the constraint on $n$ that must hold for the particle orbit to be stable under small oscillations (i.e., the mass will oscillate about the circular orbit).

   (b) For $n = 1$, compute the frequency of small radial oscillations about this circular orbit.

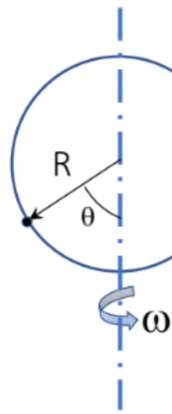
3. A passenger automobile of length $L$ (measured between the rear and front wheels) is driven off a cliff and falls by a vertical distance $H$. The automobile hits the ground at a distance $D$ away from the cliff.

   Provide an approximate expression for the the number of rotations the automobile would make before it hits the ground in terms of $D$, $H$, and $L$.

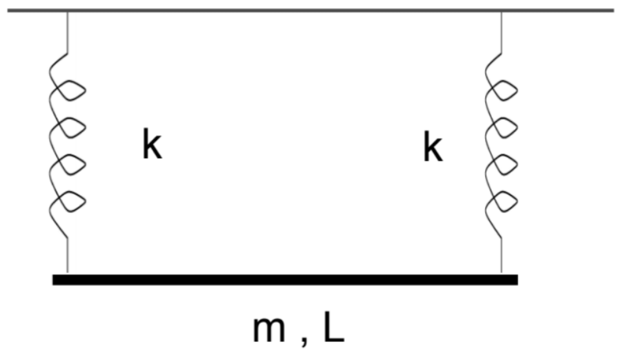
4. A circular wire hoop of radius $R$ spins with fixed angular velocity $\omega$ about a vertical axis passing through its center. A point particle of mass $m$ slides without friction along the wire. There is a gravitational acceleration $g$ in the downward direction.

   (a) Find all values of $\theta$ where the particle can be stationary with $\theta$ fixed and classify their stability for each of the stationary modes.

   (b) For the stable modes, find the frequency for small oscillations.



5. A rod of length $L$ and mass $m$ is supported by two springs of spring constant $k$ at the ends of the rod as shown in the figure. Assuming that the rod remains in the vertical plane and that there is no swinging motion in the horizontal directions, find the normal modes for small oscillations and the oscillation frequency for each mode.



Question 1 Solution

(a) Since the whole rope moves at once we can simply use energy conservation. Since the center of mass has fallen a distance $\frac{L}{2}$ when the rope loses contact with the table,

$$0 = -Mg\frac{L}{2} + \frac{1}{2}Mv_A^2$$

$$v_A = \sqrt{gL}.$$

(b) In this case only the piece of rope hanging is moving since the rope on the table is piled up and not moving. Let the density (mass per unit length) be $\lambda$ and $x$ be the amount of rope hanging through the hole at time $t$. The gravitational force on that piece is $(\lambda x)g$ which is responsible for the change in momentum $(\lambda x \dot{x})$ of that piece (the rest of the rope is not moving in this case). Thus

$$(\lambda x)g = \frac{d\,(\lambda x \dot{x})}{dt} = \lambda x \ddot{x} + \lambda \dot{x}^2$$

$$0 = x\ddot{x} + \dot{x}^2 - xg.$$

Looking at this differential equation we can guess at an answer that looks like

$$x(t) = Ct^b.$$

On dimensional grounds and noting the equation of contains $g$, and therefore $b = 2$ so

$$x(t) = Dgt^2.$$

Substituting into the differential equation

$$0 = 2D^2g^2t^2 + 4D^2g^2t^2 - Dg^2t^2$$

$$D = \frac{1}{6}.$$

So substituting in we can write

$$x(t) = \frac{1}{2} \cdot \frac{g}{3} t^2,$$

which compared to the usual kinematic corresponds to a particle with constant acceleration $a = \frac{g}{3}$, so using the usual kinematic equations for constant acceleration $(v^2 = v_0^2 + 2as)$ leads to

$$v_B = \sqrt{\frac{2}{3}gL}.$$

So since $v_A > v_B$ the the straight rope A has a large speed than rope B when the ropes lose the contact with the table.

Note: Since the problem specifies for rope B that the part on the table is not moving, this means that at the edge of the table the pieces of the rope suddenly go from stationary to moving. This is essentially an inelastic collision, so mechanical energy is not conserved. This is why using a Lagrangian will not work. This is more obvious in the discrete case, where one imagines the rope to be made of small masses joined together by massless strings. As one mass goes over the table, it will fall at some velocity until the string connected to the mass behind it suddenly goes taut, pulling the next mass over the table.

This also makes it clearer why there is no tension at the top of the rope B. In the discrete case, the string connecting the top mass and the next one is loose until the mass falls the full distance and the string goes taut, at which point the next mass on the table will be suddenly pulled off the table and become the top mass.

Question 2 Solution

(a) The effective potential is

$$V_{\text{eff}}(r) = V(r) + \frac{L^2}{2mr^2}.$$

The requirement for a circular orbit is $\frac{dV_{\text{eff}}}{dr} = 0$, i.e.,

$$\frac{kmn}{r^{n+1}} - \frac{L^2}{mr^3} = 0.$$

The requirement for a stable orbit is $\frac{d^2V_{\text{eff}}}{dr^2} > 0$, i.e.,

$$\begin{aligned}
-\frac{kmn(n+1)}{r^{n+2}} + \frac{3L^2}{mr^4} &> 0 \\
-\frac{L^2}{mr^3} \cdot \frac{(n+1)}{r} + \frac{3L^2}{mr^4} &> 0 \qquad \text{(using the previous condition)} \\
(2-n)\frac{L^2}{mr^4} &> 0 \\
n &< 2
\end{aligned}$$

(b) For $n = 1$, we find

$$\frac{km}{r^2} - \frac{L^2}{mr^3} = 0 \implies r_0 = \frac{L^2}{km^2}.$$

Let $r(t) = r_0 + \epsilon(t)$. This implies

$$m\ddot{\epsilon} + \left.\frac{d^2V_{\text{eff}}}{dr^2}\right|_{r=r_0} \epsilon = 0,$$

which yields a frequency $\omega = \sqrt{\frac{1}{m}\left.\frac{d^2V_{\text{eff}}}{dr^2}\right|_{r=r_0}}$. We compute

$$\begin{aligned}
\left.\frac{d^2V_{\text{eff}}}{dr^2}\right|_{r=r_0} &= -\frac{2km}{r_0^3} + \frac{3L^2}{mr_0^4} \\
&= -2km \cdot \frac{k^3m^6}{L^6} + \frac{3L^2}{m} \cdot \frac{k^4m^8}{L^8} \\
&= km \cdot \frac{k^3m^6}{L^6} \\
&= \frac{km}{r_0^3}.
\end{aligned}$$

Thus

$$\omega = \sqrt{\frac{k}{r_0^3}} = \frac{k^2m^3}{L^3}.$$

Question 3 Solution

The key to this problem is the estimation of angular momentum gained when the front wheels are off the cliff but the rear wheels are still supported by the ground. The rotation axis during this short period of time is the rear axle of the car (in the reference frame of the car). The duration is approximately equal to length between the wheels $l$ divided by the speed of the car $v$. The angular momentum will be

$$L = mg\frac{l}{2}\frac{l}{v} = mg\frac{l^2}{2v}.$$

We can determine the velocity of the car from the ratio of distance to the time it takes to fall

$$v = \frac{d}{\sqrt{2h/g}}$$

$$L = m\sqrt{\frac{gh}{2}}\frac{l^2}{d}.$$

Using the angular momentum we can determine the angular velocity

$$L = I\omega = \frac{1}{3}ml^2\omega.$$

Solving for $\omega$,

$$\omega = \sqrt{\frac{9gh}{2d^2}}.$$

The number of rotations can be found by multiplying the rotation rate with the time to impact:

$$n = \frac{\omega}{2\pi}\sqrt{\frac{2h}{g}} = \frac{3h}{2\pi d}.$$

Question 4 Solution

Take the zero of the potential energy to be at $\theta = \pi/2$, i.e., $V = -mgR\cos\theta$. Then the Lagrangian is

$$L = T - V = \frac{1}{2}m\left(R\dot{\theta}\right)^2 + \frac{1}{2}m\left(R\sin\theta\omega\right)^2 + mgR\cos\theta,$$

and this results in the equation of motion

$$\begin{aligned} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} &= 0 \\ \frac{d}{dt}\left(mR^2\dot{\theta}\right) - \frac{\partial L}{\partial \theta} &= 0. \end{aligned}$$

Thus we see that the stationary points occur when $\dfrac{\partial L}{\partial \theta} = 0$. We compute

$$\begin{aligned} 0 &= \frac{\partial L}{\partial \theta} \\ &= \frac{\partial}{\partial \theta}\left(\frac{1}{2}m\left(R\sin\theta\omega\right)^2 + mgR\cos\theta\right) \\ &= mR^2\omega^2\sin\theta\cos\theta - mgR\sin\theta \\ 0 &= \sin\theta\left(R\omega^2\cos\theta - g\right). \end{aligned}$$

The solutions are $\theta = 0, \pi$, and $\cos\theta = \dfrac{g}{R\omega^2}$. We now investigate the stability of the equation of motion $mR^2\ddot{\theta} = \dfrac{\partial L}{\partial \theta}$ about these points.

$\underline{\theta = 0:}$

We find

$$\begin{aligned} mR^2\ddot{\theta} &= \frac{\partial L}{\partial \theta} \\ &\approx mR^2\omega^2\theta - mgR\theta \\ \ddot{\theta} &= \left(\omega^2 - \frac{g}{R}\right)\theta. \end{aligned}$$

Thus there is simple harmonic oscillation about 0 for $\omega^2 < \dfrac{g}{R}$, but the motion is unstable for larger values of $\omega$.

$\underline{\theta = \pi:}$

Using $\sin(\pi + \delta\theta) = -\sin\delta\theta$ and $\cos(\pi + \delta\theta) = -\cos\delta\theta$, we find

$$\begin{aligned}
\delta\ddot{\theta} &= \omega^2 \sin\delta\theta \cos\delta\theta + \frac{g}{R}\sin\delta\theta \\
&\approx \omega^2 \delta\theta + \frac{g}{R}\delta\theta \\
&= \left(\omega^2 + \frac{g}{R}\right)\delta\theta.
\end{aligned}$$

Motion about this point is always unstable (as expected).

$\underline{\cos\theta_0 = \frac{g}{R\omega^2}:}$

We find

$$\begin{aligned}
\delta\ddot{\theta} &= \omega^2 \sin(\theta_0 + \delta\theta)\cos(\theta_0 + \delta\theta) - \frac{g}{R}\sin(\theta_0 + \delta\theta) \\
&= \frac{1}{2}\omega^2 \sin(2(\theta_0 + \delta\theta)) - \frac{g}{R}\sin(\theta_0 + \delta\theta) \\
&\approx \frac{1}{2}\omega^2 \sin(2\theta_0) - \frac{g}{R}\sin(\theta_0) + \left(\omega^2\cos(2\theta_0) - \frac{g}{R}\cos\theta_0\right)\delta\theta \\
&= \left(\omega^2\cos(2\theta_0) - \frac{g}{R}\cos\theta_0\right)\delta\theta \qquad \left(\left.\frac{\partial L}{\partial\theta}\right|_{\theta=\theta_0} = 0.\right) \\
&= \left(\omega^2\left(2\cos^2\theta_0 - 1\right) - \frac{g}{R}\cos\theta_0\right)\delta\theta \\
&= \left(\omega^2\left(2\frac{g^2}{R^2\omega^4} - 1\right) - \frac{g}{R}\cdot\frac{g}{R\omega^2}\right)\delta\theta \\
&= \left(\left(\frac{g}{R\omega}\right)^2 - \omega^2\right)\delta\theta.
\end{aligned}$$

Motion about this point is stable for $\omega^2 > \frac{g}{R}$.

Thus we see that for $\omega < \sqrt{\frac{g}{R}}$, there is one stable stationary point at $\theta = 0$, and oscillations around it have frequency $\sqrt{\frac{g}{R} - \omega^2}$. For $\omega > \sqrt{\frac{g}{R}}$, there is one stable point at $\theta = \cos^{-1}\left(\frac{g}{R\omega^2}\right)$, and oscillations around it have frequency $\sqrt{\omega^2 - \left(\frac{g}{R\omega}\right)^2}$.

Question 5 Solution

Use the vertical distance $x$ as one generalized coordinate and the rotation angle $\theta$ with respect to the rod's center of mass as the other generalized coordinate. These turn out to be the normal mode coordinates since the equations are not coupled.

We find

$$T = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}I\dot{\alpha}^2, \qquad U = kx^2 + \frac{kL^2}{4}\alpha^2,$$

where $I = \dfrac{mL^2}{12}$.

The $x$ equation is

$$m\ddot{x} + 2kx = 0,$$

so the frequency of oscillations $\omega = \sqrt{\dfrac{2k}{m}}$.

The $\alpha$ equation is

$$I\ddot{\alpha} + \frac{kL^2}{2}\alpha = 0,$$

so the frequency of oscillations is $\omega = \sqrt{\dfrac{kL^2}{2I}} = \sqrt{\dfrac{6k}{m}}$.