Columbia University
Department of Physics
QUALIFYING EXAMINATION

Wednesday, January 13, 2016
1:00PM to 3:00PM
Modern Physics
Section 3. Quantum Mechanics

Two hours are permitted for the completion of this section of the examination. Choose 4 problems out of the 5 included in this section. (You will not earn extra credit by doing an additional problem). Apportion your time carefully.

Use separate answer booklet(s) for each question. Clearly mark on the answer booklet(s) which question you are answering (e.g., Section 3 (QM), Question 2, etc.).

Do **NOT** write your name on your answer booklets. Instead, clearly indicate your **Exam Letter Code**.

You may refer to the single handwritten note sheet on $8\frac{1}{2}$" × 11" paper (double-sided) you have prepared on Modern Physics. The note sheet cannot leave the exam room once the exam has begun. This note sheet must be handed in at the end of today's exam. Please include your Exam Letter Code on your note sheet. No other extraneous papers or books are permitted.

Simple calculators are permitted. However, the use of calculators for storing and/or recovering formulae or constants is NOT permitted.

Questions should be directed to the proctor.

Good Luck!

1. The ground-state wave function of the hydrogen atom is $\psi_{1s}(r) \sim e^{-r/a_0}$, where the Bohr radius $a_0 = 5.29 \times 10^{-11}$ m. This solution, and the corresponding ground-state energy

$$E_0 = -\frac{e^2}{2a_0} = -13.6 \text{ eV},$$

is derived assuming that the proton is a point charge. In reality the proton's charge is distributed over its radius R, which we take to be 10^{-15} m.

   (a) Assuming that the proton's charge is uniformly distributed over a solid sphere of radius R, estimate the fractional shift $\Delta E/|E_0|$ in the ground-state energy of the hydrogen atom from the value obtained assuming the proton is a point charge. [Computing the shift to lowest non-vanishing order in the small parameter R/a_0 suffices.]

   (b) How would the order of magnitude of your answer change if the electron in the hydrogen atom were replaced with a negative muon, with mass $m_\mu = 207\, m_e$?

2. Consider a particle in one dimension with a potential energy

$$V(x) = \begin{cases} -U\,, & -c < x < c \\ 0\,, & \text{otherwise} \end{cases}$$

where U is a positive constant.

(a) Consider the wave function

$$\psi(x) = \begin{cases} a(b+x)\,, & -b < x < 0 \\ a(b-x)\,, & 0 < x < b \\ 0\,, & \text{otherwise} \end{cases}$$

where a and b are constants and $b > c$. What is the expectation value of the Hamiltonian in this state?

(b) Use the result from part (a) to show that there will be a bound state for any value of U.

(c) Use the result from part (b) to show that a one-dimensional potential energy that is (i) equal to zero at $x = \pm\infty$ (ii) nowhere greater than zero, and (iii) less than zero in some finite interval, always has a bound state.

3. Consider a particle in one dimension with an *inverted* (upside-down) harmonic oscillator potential $V(x) = -\frac{1}{2}m\omega^2 x^2$. Picking units such that $m = 1$, $\hbar = 1$, the Schrödinger equation takes the form:

$$i\partial_t\psi = -\tfrac{1}{2}\partial_x^2\psi - \tfrac{1}{2}\omega^2 x^2\psi$$

Due to the minus sign in the potential, this describes an unstable system, with energy unbounded from below.

(a) Consider a state at time $t = 0$ given by a Gaussian wave packet of the form

$$\psi_0(x) = \alpha_0\, e^{-\beta_0 x^2}\,, \qquad \beta_0 \equiv \frac{\omega}{2}\tan\theta\,.$$

Here α_0, β_0 and θ are real constants, with the parametrization of β_0 in terms of θ introduced for future convenience. Show that the wave function evolves in time as

$$\psi(x,t) = \alpha(t)\, e^{-\beta(t)\, x^2}\,,$$

and find $\beta(t)$ explicitly.

(b) Find the late time ($t \to \infty$) behavior of $\beta(t)$ and use this to show that at late times, the expectation value of $\hat{x}^2$ is $\langle \hat{x}^2 \rangle \propto e^{2\omega t}$. What is the proportionality constant? Here $\hat{x}$ is the position operator.

(c) Show similarly that at late times, the expectation value $\langle (\hat{p} - \omega\hat{x})^2 \rangle$ decays exponentially. What is the exponent? Here $\hat{p}$ is the momentum operator. This exponential decay is the signature of a squeezed state.

4. Assume that a particle is moving in an harmonic oscillator potential, $V(x) = \frac{1}{2}m\omega^2 x^2$. At an initial time, say $t = 0$, we are given that its wave function is

$$\psi(x,0) = N \sum_n \left(\tfrac{1}{\sqrt{7}}\right)^n \psi_n(x) \,,$$

where the $\psi_n(x)$ are the usual orthonormal energy eigenstates of the harmonic oscillator.

(a) Find the value of the normalization constant N.

(b) Show that the probability of finding the particle at a given position x is a periodic function of t, and find the period.

(c) Find the expectation value of the energy.

5. Consider the two-dimensional isotropic harmonic oscillator, picking units such that $m = 1$, $\omega = 1$ and $\hbar = 1$, so the Hamiltonian is

$$H = \frac{1}{2}(p_x^2 + p_y^2) + \frac{1}{2}(x^2 + y^2) \ .$$

with $[x, p_x] = i = [y, p_y]$. Interpreting the system as being made up of two independent one-dimensional oscillators, the energy spectrum can be generated using the standard one-dimensional creation and annihilation operators

$$a_x = \frac{1}{\sqrt{2}}(x + ip_x) \ , \qquad a_x^\dagger \ , \qquad a_y = \frac{1}{\sqrt{2}}(y + ip_y) \ , \qquad a_y^\dagger \ ,$$

in terms of which $H = a_x^\dagger a_x + a_y^\dagger a_y + 1$, and $[H, a_x] = -a_x$, $[H, a_x^\dagger] = a_x^\dagger$, etc.

(a) Construct the energy spectrum using these operators, and find the degeneracy of each energy level.

(b) Consider now the angular momentum operator,

$$L = xp_y - yp_x \ .$$

Express L in terms of the a, $a^\dagger$, and show that the basis of energy eigenstates constructed above does *not* diagonalize the angular momentum operator.

(c) Find a new basis of creation and annihilation operators, obtained as linear combinations of the one-dimensional operators defined above, which generate a basis of energy eigenstates that *does* diagonalize the angular momentum operator. (Hint: You might make use of an analogy with the relation between linear and circular polarization.) What are the possible values of the angular momentum in each energy level?

1 Zajc: Finite Size Correction in H-atom

1.1 Problem

The ground-state wave function of the hydrogen atom is $\psi_{1s}(r) \sim e^{-r/a_0}$, where the Bohr radius $a_0 = 5.29 \times 10^{-11}$ m. This solution, and the corresponding ground-state energy

$$E_0 = -\frac{e^2}{2a_0} = -13.6 \text{ eV},$$

is derived assuming that the proton is a point charge. In reality the proton's charge is distributed over its radius R, which we take to be 10^{-15} m.

a) Assuming that the proton's charge is uniformly distributed over a solid sphere of radius R, estimate the fractional shift $\Delta E/|E_0|$ in the ground-state energy of the hydrogen atom from the value obtained assuming the proton is a point charge. [Computing the shift to lowest non-vanishing order in the small parameter R/a_0 suffices.]

b) How would the order of magnitude of your answer change if the electron in the hydrogen atom were replaced with a negative muon, with mass $m_\mu = 207\ m_e$?

1.2 Solution

a) Gauss's law tells us field and potential are modified only for $r < R$, where the field is linearly increasing with r. Requiring the potential calculated from this field to match at $r = R$ gives

$$\phi(r) = \begin{cases} -\frac{er^2}{2R^3} + \frac{3}{2}\frac{e}{R} & r \le R \\ +\frac{e}{r} & r \ge R \end{cases} \tag{1.1}$$

The potential energy is then $V(r) = -e\phi(r)$, and the perturbation potential $V_p(r) = V(r) - (-\frac{e^2}{r})$ has support only in $r \le R$:

$$V_p(r) = \frac{e^2}{2R}\left[\left(\frac{r}{R}\right)^2 - 3\right] + \frac{e^2}{r} \quad \text{if } r \le R \quad .$$

Then, to lowest order in perturbation theory

$$\Delta E = \frac{\langle\psi_{1s}|V_p|\psi_{1s}\rangle}{\langle\psi_{1s}|\psi_{1s}\rangle} = \frac{\int_0^\infty e^{-2r/a_o} V_p(r)\, 4\pi r^2 dr}{\int_0^\infty e^{-2r/a_o}\, 4\pi r^2 dr} = \frac{\int_0^R e^{-2r/a_o} V_p(r)\, r^2 dr}{\int_0^\infty e^{-2r/a_o}\, r^2 dr}$$

Since $R \ll a_0$, the wave-function varies very little over the range of $V_p(r)$, so we can approximate the above expression to leading order in the small parameter $\frac{R}{a_0} \sim 10^{-5}$ as

$$\Delta E = \frac{\int_0^R V_p(r)\, r^2 dr}{\int_0^\infty e^{-2r/a_o}\, r^2 dr} \quad .$$

Then

$$\int_0^\infty e^{-2r/a_o}\, r^2 dr = \left(\frac{a_0}{2}\right)^3 2! = \frac{a_0^3}{4}$$

and

$$\begin{aligned}\Delta E = \frac{4}{a_0^3}\int_0^R V_p(r)\, r^2 dr &= \frac{4}{a_0^3}\int_0^R \left(\frac{e^2}{2R}\left[\left(\frac{r}{R}\right)^2 - 3\right] + \frac{e^2}{r}\right) r^2 dr \\ &= \frac{4e^2}{a_0^3}\left(\frac{1}{10}R^2 - \frac{3}{2R}\frac{R^3}{3} + \frac{R^2}{2}\right) \\ &= \frac{2}{5}\frac{e^2R^2}{a_0^3} = \frac{4}{5}\left(\frac{R}{a_0}\right)^2\frac{e^2}{2a_0} = \frac{4}{5}\left(\frac{R}{a_0}\right)^2 |E_0|\,.\end{aligned}$$

We conclude that to leading order

$$\frac{\Delta E}{|E_0|} = \frac{4}{5}\left(\frac{R}{a_0}\right)^2 = \boxed{3 \times 10^{-10}}.$$

The correction is positive, that is, the state is slightly less bound due to distribution of the proton's charge. It is tiny, less than a part per billion. Taking into account higher orders in perturbation theory would lead to even tinier corrections, suppressed by powers of the small parameter $\frac{R}{a_0} \sim 10^{-5}$.

As a check of our result, notice it is consistent with the simple estimate:

$$\Delta E \sim \text{(probability electron in affected region)} \times \text{(scale of average potential energy shift)}$$
$$\sim \frac{R^3}{a_o^3} \times \frac{e^2}{R} \sim \frac{R^2}{a_o^2}\,|E_0|\,.$$

b) For the case of muonic hydrogen, the Bohr radius decreases by a factor of about 200, since

$$a_0 = \frac{\hbar^2}{m_e e^2} \to \frac{\hbar^2}{m_\mu e^2} \quad .$$

(Students should be able to derive this very quickly from simple Bohr model for the system. Alternatively they can easily infer the parametric dependence of the atom's scale a by finding the minimum of $H(a) \sim \frac{\hbar^2}{2ma^2} - \frac{e^2}{a}$, which can be viewed as a crude variational estimate.) That means that the fractional shift in the ground-state energy increases by 200^2, so

$$3 \times 10^{-10} \to 200^2 \times 3 \times 10^{-10} \sim \boxed{10^{-5}}\,.$$

Note also that the ground-state energy increases in magnitude by a factor of 200.

Note: The above estimate assumed effectively that the proton has infinite mass. For the proton-electron system this is an excellent approximation since $m_e/m_p \approx 5.4\times10^{-4}$, but for the proton-muon system we have $m_\mu/m_p \approx 0.11$, which is more marginal. It is still good enough to estimate the order of magnitude, which is what was asked. However if we wanted to work at the same level of precision as in the first part of this problem, we should actually consider the change in *reduced* mass of the system rather than just that of the negative particle. In general, if we have a Coulomb system consisting of a particle of mass M interacting with a particle of mass m, we may factor out the center of mass to obtain a relative Hamiltonian, which is of single particle form but with a reduced effective mass $\mu = \frac{Mm}{M+m} = \frac{m}{1+\frac{m}{M}}$. Taking this into account we get $\psi_{1s}(r) \sim e^{-r/a}$ where $a = \frac{\hbar^2}{\mu e^2} = \frac{m}{\mu}\,a_\infty = (1+\frac{m}{M})a_\infty$, with a_∞ the value of a in the limit $M \to \infty$. Thus for the proton-electron atom we have $a_\infty = a_o$ and $a \approx 1.00054\,a_o$, which is for our purposes certainly a negligible correction. For the proton-muon atom we have $a \approx 1.11\,a_\infty \approx 1.11\,a_o/207 \approx a_o/186$. So $\Delta E/|E_0|$ for the muon actually gets rescaled by a factor $186^2 \approx 3.5\times10^4$, instead of the naive $207^2 \approx 4.3\times10^4$. The relative difference is about 25%, which is of the same order of magnitude as the differences in geometric factors one gets for different models of the proton charge distribution (e.g. solid vs hollow sphere).

3 Weinberg II: Existence of Bound States

3.1 Problem

Consider a particle in one dimension with a potential energy

$$V(x) = \begin{cases} -U\,, & -c < x < c \\ 0\,, & \text{otherwise} \end{cases}$$

where U is a positive constant.

(a) Consider the wave function

$$\psi(x) = \begin{cases} a(b+x)\,, & -b < x < 0 \\ a(b-x)\,, & 0 < x < b \\ 0\,, & \text{otherwise} \end{cases}$$

where a and b are constants and $b > c$. What is the expectation value of the Hamiltonian in this state?

(b) Use the result from part (a) to show that there will be a bound state for any value of U.

(c) Use the result from part (b) to show that a one-dimensional potential energy that is (i) equal to zero at $x = \pm\infty$ (ii) nowhere greater than zero, and (iii) less than zero in some finite interval, always has a bound state.

3.2 Solution

Solution to Weinberg 2-2016

a) $$\langle H \rangle = \frac{\int dx\, \psi^* H \psi}{\int dx\, \psi^* \psi}$$

$$\int dx\, \psi^* \psi = 2\int_0^b dx\, a^2 (b-x)^2 = \frac{2}{3} a^2 b^3$$

$$\int dx\, \psi^* H \psi = \int dx \left\{ -\frac{\hbar^2}{2m} \psi^* \psi'' + V \psi^* \psi \right\}$$

$$-\frac{\hbar^2}{2m}\int dx\, \psi^* \psi'' = \frac{\hbar^2}{2m}\int_{-b}^{b} dx\, (\psi')^2 = \frac{\hbar^2}{m} b a^2$$

$$\int dx\, \psi^* \psi V = -2U\int_0^c dx\, a^2 (b-x)^2 = -\frac{2U}{3} a^2 \left[(b-c)^2 - b^3 \right]$$

$$\langle H \rangle = \frac{3}{2a^2 b^3} \left\{ \frac{\hbar^2}{m} b a^2 - \frac{2U}{3} a^2 (3b^2 c - 3bc^2 + c^3) \right\}$$

$$= \frac{3}{2} \left\{ \frac{\hbar^2}{m} \frac{1}{b^2} - \frac{2U}{3} \left(\frac{3c}{b} - \frac{3c^2}{b^2} + \frac{c^3}{b^3} \right) \right\}$$

b) For $b >> c$, $\langle H \rangle \approx -\frac{3Uc}{b} < 0$,

so there must be a energy eigenstate with $E < 0$

c) For any $V(x)$ satisfying these conditions, one can find a $\tilde{V}(x)$ of the form given in this problem such that $V(x) \le \tilde{V}(x)$ for all x, so the energy eigenvalue $E \le \tilde{E} < 0$

4 Hui: Time Evolution of Inverted Oscillator

4.1 Problem

Consider a particle in one dimension with an *inverted* (upside-down) harmonic oscillator potential $V(x) = -\frac{1}{2}m\omega^2 x^2$. Picking units such that $m = 1$, $\hbar = 1$, the Schrödinger equation takes the form:

$$i\partial_t\psi = -\tfrac{1}{2}\partial_x^2\psi - \tfrac{1}{2}\omega^2 x^2\psi$$

Due to the minus sign in the potential, this describes an unstable system, with energy unbounded from below.

1. Consider a state at time $t = 0$ given by a Gaussian wave packet of the form

   $$\psi_0(x) = \alpha_0\, e^{-\beta_0 x^2}\,, \qquad \beta_0 \equiv \frac{\omega}{2}\tan\theta\,.$$

   Here α_0, β_0 and θ are real constants, with the parametrization of β_0 in terms of θ introduced for future convenience. Show that the wave function evolves in time as

   $$\psi(x,t) = \alpha(t)\, e^{-\beta(t)\, x^2}\,,$$

   and find $\beta(t)$ explicitly.

2. Find the late time ($t \to \infty$) behavior of $\beta(t)$ and use this to show that at late times, the expectation value of $\hat{x}^2$ is $\langle\hat{x}^2\rangle \propto e^{2\omega t}$. What is the proportionality constant? Here $\hat{x}$ is the position operator.

3. Show similarly that at late times, the expectation value $\langle(\hat{p} - \omega\hat{x})^2\rangle$ decays exponentially. What is the exponent? Here $\hat{p}$ is the momentum operator. This exponential decay is the signature of a squeezed state.

4.2 Solution

1. Substitute $\psi = \alpha e^{-\beta x^2}$ into the Schrodinger equation. The terms proportional to x^2 give

$$i\frac{d\beta}{dt} = \frac{\omega^2}{2} + 2\beta^2 \tag{4.1}$$

This can be integrated to give

$$-2it = \int \frac{d\beta}{\beta^2 + \frac{1}{4}\omega^2} = \frac{2}{\omega}\left[\tan^{-1}\big(2\beta(t)/\omega\big) - \tan^{-1}\big(2\beta(0)/\omega\big)\right] . \tag{4.2}$$

Therefore, we have

$$\boxed{\beta(t) = \frac{\omega}{2}\tan\left(\theta - i\omega t\right)} . \tag{4.3}$$

(Alternatively, this can be obtained by substituting $\beta(t) = \frac{\omega}{2}\tan\theta(t)$ into the differential equation, an ansatz indirectly suggested in the formulation of the problem.)

2. In the late time limit, we see that

$$\beta(t) = -\frac{i\omega}{2} + i\omega e^{-2\omega t}\cos 2\theta + \omega e^{-2\omega t}\sin 2\theta + O(e^{-4\omega t}) . \tag{4.4}$$

The probability $|\psi|^2 \propto e^{-2(\operatorname{Re}\beta)x^2}$ is a Gaussian, and therefore

$$\langle \hat{x}^2 \rangle = \frac{1}{4\operatorname{Re}\beta} = \boxed{\frac{1}{4\omega\sin 2\theta}\, e^{2\omega t}} . \tag{4.5}$$

The wave-function spreads, as expected.

3. Observe that

$$(-i\partial_x - \omega x)\psi = (2i\beta - \omega)x\psi . \tag{4.6}$$

Therefore,

$$\langle (\hat{p} - \omega\hat{x})^2 \rangle = |2i\beta - \omega|^2 \langle \hat{x}^2 \rangle = \boxed{\frac{\omega}{\sin 2\theta}\, e^{-2\omega t}} . \tag{4.7}$$

In other words, both momentum and position spread, but this particular combination is actually squeezed.

5 Greene: Time Evolution of Oscillator

5.1 Problem

Assume that a particle is moving in an harmonic oscillator potential, $V(x) = \frac{1}{2}m\omega^2x^2$. At an initial time, say $t = 0$, we are given that its wave function is

$$\psi(x,0) = N\sum_n \left(\tfrac{1}{\sqrt{7}}\right)^n \psi_n(x)\,,$$

where the $\psi_n(x)$ are the usual orthonormal energy eigenstates of the harmonic oscillator.

1. Find the value of the normalization constant N.
2. Show that the probability of finding the particle at a given position x is a periodic function of t, and find the period.
3. Find the expectation value of the energy.

5.2 Solution

1. Using the orthonormality of the eigenfuctions, $1 = N^2 \sum_n \frac{1}{7^n} = N^2 \frac{7}{6}$, yielding $\boxed{N = \sqrt{\frac{6}{7}}}$.

2. $\psi(x,t) = e^{-iHt/\hbar}\,\psi(x,0) = \sqrt{\frac{6}{7}} \sum_n (\frac{1}{\sqrt{7}})^n e^{-i\omega t(n+\frac{1}{2})}\psi_n(x)$. Thus, the probability of finding the particle at x at time t is given by $|\psi(x,t)|^2 = \frac{6}{7} \sum_{n,m} (\frac{1}{7})^{\frac{n+m}{2}} e^{-i\omega t(n-m)}\psi_n(x)\psi_m^*(x)$. Each term in the sum is periodic in time, and all the periods are commensurate with the longest being $\boxed{T = \frac{2\pi}{\omega}}$.

3. The probability of finding the system in the n-th energy eigenstate is $p_n = \frac{6}{7} \cdot \frac{1}{7^n}$, hence the expectation value of the energy is

$$\langle \psi | H | \psi \rangle = \sum_n p_n \big(n + \tfrac{1}{2}\big)\hbar\omega = \big(\tfrac{1}{6} + \tfrac{1}{2}\big)\hbar\omega = \boxed{\tfrac{2}{3}\hbar\omega}\,.$$

6 Nicolis: Spectrum of Two-dimensional Oscillator

6.1 Problem

Consider the two-dimensional isotropic harmonic oscillator, picking units such that $m = 1$, $\omega = 1$ and $\hbar = 1$, so the Hamiltonian is

$$H = \frac{1}{2}(p_x^2 + p_y^2) + \frac{1}{2}(x^2 + y^2) \ .$$

with $[x, p_x] = i = [y, p_y]$. Interpreting the system as being made up of two independent one-dimensional oscillators, the energy spectrum can be generated using the standard one-dimensional creation and annihilation operators

$$a_x = \frac{1}{\sqrt{2}}(x + ip_x) \ , \qquad a_x^\dagger \ , \qquad a_y = \frac{1}{\sqrt{2}}(y + ip_y) \ , \qquad a_y^\dagger \ ,$$

in terms of which $H = a_x^\dagger a_x + a_y^\dagger a_y + 1$, and $[H, a_x] = -a_x$, $[H, a_x^\dagger] = a_x^\dagger$, etc.

1. Construct the energy spectrum using these operators, and find the degeneracy of each energy level.

2. Consider now the angular momentum operator,

   $$L = xp_y - yp_x \ .$$

   Express L in terms of the a, $a^\dagger$, and show that the basis of energy eigenstates constructed above does *not* diagonalize the angular momentum operator.

3. Find a new basis of creation and annihilation operators, obtained as linear combinations of the one-dimensional operators defined above, which generate a basis of energy eigenstates that *does* diagonalize the angular momentum operator. (Hint: You might make use of an analogy with the relation between linear and circular polarization.) What are the possible values of the angular momentum in each energy level?

6.2 Solution

1. The ground state $|0\rangle$ is the state annihilated by a_x and a_y. The excited states are $|n_x, n_y\rangle \propto (a_x^\dagger)^{n_x}(a_y^\dagger)^{n_y}|0\rangle$, with energies

$$\boxed{E_{n_x,n_y} = n_x + n_y + 1}\,.$$

The degeneracy of the energy level with energy $E = n+1$ equals $\boxed{n+1}$.

2. Using $x = \frac{1}{\sqrt{2}}(a_x + a_x^\dagger)$, $p_x = -\frac{i}{\sqrt{2}}(a_x - a_x^\dagger)$ and likewise for y, p_y, we get

$$\boxed{L = i(a_x a_y^\dagger - a_x^\dagger a_y)}\,,$$

The ground state has zero angular momentum, $L|0\rangle = 0$. However, $[L, a_x^\dagger] = i a_y^\dagger$ and $[L, a_y^\dagger] = -i a_x^\dagger$. This implies for example that $\boxed{L|1,0\rangle \propto |0,1\rangle}$, and thus the basis constructed above does not diagonalize L.

3. To cure this, we need to find new creation operators which are angular momentum eigenoperators, that is to say, which commute with L to a multiple of themselves:

$$[L, a_\lambda^\dagger] = \lambda\, a_\lambda^\dagger\,, \qquad a_\lambda^\dagger \equiv u a_x^\dagger + v a_y^\dagger\,.$$

Using $[L, u a_x^\dagger + v a_y^\dagger] = i u a_y^\dagger - i v a_x^\dagger$, this requirement leads to the equations $iu = \lambda v$, $-iv = \lambda u$, hence $\lambda = \pm 1$ with $v = \pm i u$. Normalizing the new operators so they obey standard canonical commutation relations, we obtain

$$\boxed{a_\pm^\dagger = \frac{1}{\sqrt{2}}(a_x^\dagger \pm i a_y^\dagger)}\,, \qquad \boxed{a_\pm = \frac{1}{\sqrt{2}}(a_x \mp i a_y)}\,,$$

satisfying $[a_+, a_+^\dagger] = 1$ etc, as well as

$$[L, a_\pm^\dagger] = \pm a_\pm^\dagger\,, \qquad [L, a_\pm] = \mp a_\pm\,.$$

The ground state $|0\rangle$ is annihilated by the annihilation operators $a_\pm$, and the excited states are

$$|n_+, n_-\rangle \propto (a_+^\dagger)^{n_+}(a_-^\dagger)^{n_-}|0\rangle\,.$$

Since $[H, a_\pm^\dagger] = a_\pm^\dagger$ and $[L, a_\pm^\dagger] = \pm a_\pm^\dagger$, these are eigenstates of both H and L with eigenvalues

$$\boxed{E_{n_+,n_-} = n_+ + n_- + 1}\,, \qquad \boxed{L_{n_+,n_-} = n_+ - n_-}\,.$$

The possible values of L in the energy level with $E = n+1$ are

$$\boxed{L = -n + 2k\,, \quad k = 0, 1, \ldots, n}\,.$$