Columbia University
Department of Physics
QUALIFYING EXAMINATION

Wednesday, January 16, 2013
1:00PM to 3:00PM
Modern Physics
Section 3. Quantum Mechanics

Two hours are permitted for the completion of this section of the examination. Choose 4 problems out of the 5 included in this section. (You will not earn extra credit by doing an additional problem). Apportion your time carefully.

Use separate answer booklet(s) for each question. Clearly mark on the answer booklet(s) which question you are answering (e.g., Section 3 (QM), Question 2, etc.).

Do **NOT** write your name on your answer booklets. Instead, clearly indicate your **Exam Letter Code**.

You may refer to the single handwritten note sheet on $8\frac{1}{2}$" × 11" paper (double-sided) you have prepared on Modern Physics. The note sheet cannot leave the exam room once the exam has begun. This note sheet must be handed in at the end of today's exam. Please include your Exam Letter Code on your note sheet. No other extraneous papers or books are permitted.

Simple calculators are permitted. However, the use of calculators for storing and/or recovering formulae or constants is NOT permitted.

Questions should be directed to the proctor.

Good Luck!

1. Consider a spinless particle in some spherically symmetric potential $V(r)$. The Hamiltonian $\hat{H} = \frac{1}{2m}\hat{\vec{p}}^2 + V(r) \equiv \hat{T} + \hat{V}$ is thus the sum of the kinetic and potential energies.

   (a) Show that the expectation value of the commutator of any observable $\hat{A}$ and the Hamiltonian vanishes for any stationary state (any eigenvector of the Hamiltonian). Using the fact for $\hat{A} = \hat{\vec{x}}\hat{\vec{p}} + \hat{\vec{p}}\hat{\vec{x}}$ derive the identity known as the virial theorem:

   $$2\left\langle \hat{T} \right\rangle_{E,\alpha} = \Sigma_i \left\langle \hat{x}_i \frac{\partial V}{\partial x_i} \right\rangle_{E,\alpha} = \left\langle r \frac{dV}{dr} \right\rangle_{E,\alpha} \tag{1}$$

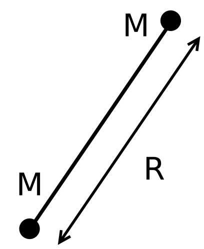
   Show your work.

   (b) Consider the potential $V(r) = \hbar^2\gamma/2mr^2$, where γ is a dimensionless parameter. Assume momentarily the possibility of bound states and use the virial theorem to calculate the expectation value of the energy. What would be your conclusion?

   (c) Now consider the modified potential $V(r) = \hbar^2\gamma/2mr^{2-\eta}$ with $0 < \eta < 1$. What are the units of γ? Using dimensional analysis, estimate the characteristic length (size of the bound state) and characteristic energy (energy of the ground state).

   (d) Try to interpret the results of b) and c).

2. Two point particles of mass M and charges $\pm Q$ are joined by a massless rod of total length R which is free to move in three dimensions.



   (a) Find the energy eigenvalues of this quantum system.

   (b) Find the four lowest normalized energy eigenstates of the system which have zero spatial momentum. Specify their normalization.

   (c) If a uniform electric field $\vec{E}$ is applied to the system find the shift in the ground state energy to first and second order in E.

   (d) Is the energy of the ground state increased or decreased by the presence of the electric field?

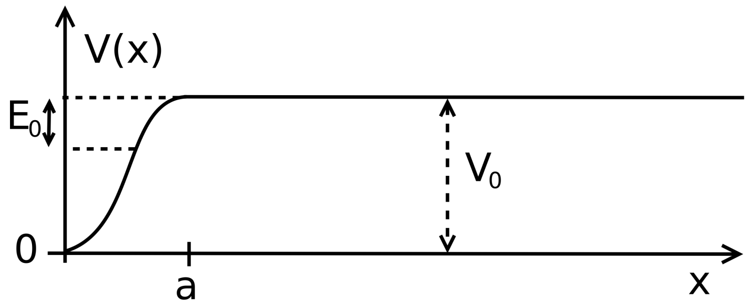
3. Consider a two-level quantum system described by Hamiltonian H, with states $|a\rangle$ and $|b\rangle$ and energies $E_a = 0$ and $E_b = E_0$. The system is initially in state $|a\rangle$. Suppose that a constant perturbation H' is applied from time $t = 0$ until some arbitrary subsequent time t.

   Find the probability $P_b(t)$ of being in state $|b\rangle$ just after the perturbation has been removed. Sketch the variation with time t, indicating the characteristic time scales on the plot. (Partial credit will be given for an accurate plot, even if the information is incomplete.) The problem should be solved *without approximation* for two forms of the perturbation H':

   (a) $\langle a| H' |a\rangle = U_a$, $\langle b| H' |b\rangle = U_b$, $\langle a| H' |b\rangle = \langle b| H' |a\rangle = 0$

   (b) $\langle a| H' |a\rangle = 0$, $\langle b| H' |b\rangle = 0$, $\langle a| H' |b\rangle = \langle b| H' |a\rangle = U$

4. A one-dimensional attractive potential well, $V(x)$, binds a mass m particle to a reflecting wall. The binding energy is $-|E_0|$ relative to $V(x)$ at large distances away.



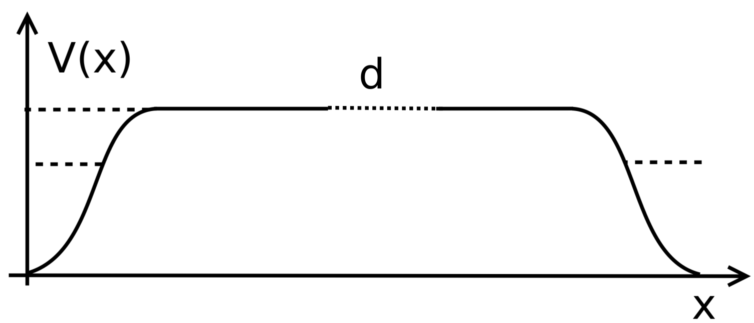
$$V(x) = V_0 \text{ for } x > a, \quad (2)$$
$$= 0 \text{ for } x = 0 \quad (3)$$

For $x \gtrsim a$ the particle wavefunction is $\psi_0 = ke^{-\alpha x}$.

(a) What is α?

(b) Estimate k if the probability for the particle being inside and outside the potential well are comparable.

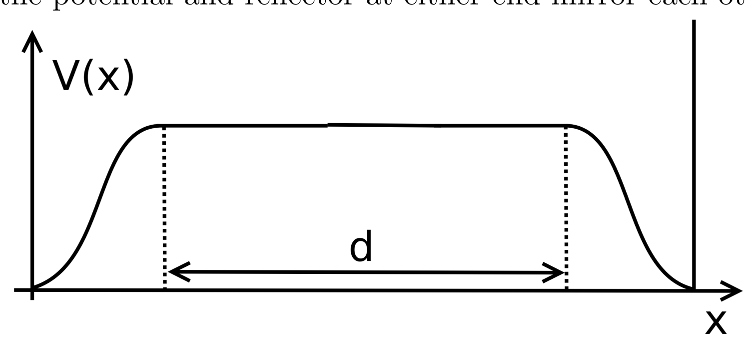
Now suppose that after a long interval d ($d \gg a$), $V(x)$ drops back to its value at $x = 0$.



If at time $t = 0$ the particle is in the state with wavefunction ψ_0 the probability for still finding it near the wall at later times is a diminishing function of time $P(t) \sim e^{-t/\tau}$.

(c) Estimate τ.

A reflecting wall is now also inserted at a distance $x = a + d + a$ from the first one at $x = 0$ so that the potential and reflector at either end mirror each other.



(d) What is the new ground state from combining $\psi_0(x)$ and $\psi_0(d-x)$?

(e) Estimate its tiny energy shift, δE, relative to $-|E_0|$. (Assume that $\psi_0 = ke^{-\alpha x}$ is an adequate approximation for all x.)

(f) What is the new $P(t)$? How does the time for it to drop to, say, $\frac{1}{2}$ compare to that for c)?

5. Two spin-1/2 particles are known to be in the singlet state. Let $S_a^{(1)}$ be the component of the spin angular momentum of particle 1 in the direction defined by the unit vector $\hat{\mathbf{a}}$, and $S_b^{(2)}$ be the component of the spin angular momentum of particle 2 in the direction defined by the unit vector $\hat{\mathbf{b}}$. Determine the expectation value of $S_a^{(1)} S_b^{(2)}$ as a function of the angle θ between $\hat{\mathbf{a}}$ and $\hat{\mathbf{b}}$.

Quantum Mechanics 2012

Virial theorem and falling on the center.

Consider a spinless particle in some spherically symmetric potential *V(r)*. The Hamiltonian $\hat{H} = \frac{1}{2m}\hat{\vec{p}}^2 + V(r) \equiv \hat{T} + \hat{V}$ is thus the sum of the kinetic and potential energies.

a) Show that the expectation value of the commutator of any observable $\hat{A}$ and the Hamiltonian vanishes for any stationary state (any eigenvector of the Hamiltonian). Using this fact for $\hat{A} = \hat{\vec{x}}\hat{\vec{p}} + \hat{\vec{p}}\hat{\vec{x}}$ derive the identity known as the virial theorem:

$$2\left\langle \hat{T} \right\rangle_{E,\alpha} = \sum_i \left\langle \hat{x}_i \frac{\partial V}{\partial x_i} \right\rangle_{E,\alpha} = \left\langle r \frac{dV}{dr} \right\rangle_{E,\alpha}$$

Show your work.

b) Consider the potential $V(r) = \hbar^2\gamma / 2mr^2$, where γ is a dimensionless parameter. Assume momentarily the possibility of bound states and use the virial theorem to calculate the expectation value of the energy. What would be your conclusion?

c) Now consider the modified potential $V(r) = \hbar^2\gamma / 2mr^{2-\eta}$ with $0 < \eta < 1$. What are the units of γ ? Using dimension analysis estimate the characteristic length (size of the bound state) and characteristic energy (energy of the ground state)

d) Try to interpret the results of b) and c)

Solution:

a) Let $|E,\alpha\rangle$ be a stationary state with the energy *E*.

$$\langle E,\alpha | \left[\hat{H},\hat{A}\right] | E,\alpha\rangle = \langle E,\alpha | \hat{H}\hat{A} - \hat{A}\hat{H} | E,\alpha\rangle =$$
$$= E\langle E,\alpha|\hat{A}|E,\alpha\rangle - E\langle E,\alpha|\hat{A}|E,\alpha\rangle = 0 .$$

For $\hat{A} = \hat{\vec{x}}\hat{\vec{p}} + \hat{\vec{p}}\hat{\vec{x}}$ and $\hat{H} = \frac{1}{2m}\hat{\vec{p}}^2 + V(\vec{x})$ the commutator $\left[\hat{H},\hat{A}\right]$ equals to

$$\left[\hat{H}, \hat{\vec{x}}\hat{\vec{p}} + \hat{\vec{p}}\hat{\vec{x}}\right] = \left[\frac{1}{2m}\hat{\vec{p}}^2 + V(\vec{x}), \hat{\vec{x}}\hat{\vec{p}} + \hat{\vec{p}}\hat{\vec{x}}\right] = -\frac{2i\hbar}{m}\hat{\vec{p}}^2 + 2i\hbar\frac{\partial V}{\partial \vec{x}}\vec{x}$$

Therefore

$$0=\left\langle -\frac{2i\hbar}{m}\hat{\vec{p}}^2+2i\hbar\frac{\partial V}{\partial\vec{x}}\vec{x}\right\rangle_{E,\alpha} \quad\Rightarrow\quad \left\langle\frac{\hat{\vec{p}}^2}{m}\right\rangle_{E,\alpha} = \left\langle\frac{\partial V}{\partial\vec{x}}\vec{x}\right\rangle_{E,\alpha}.$$

This can be rewritten in the form of the virial theorem

$$2\left\langle\hat{T}\right\rangle_{E,\alpha}=\sum_i\left\langle\hat{x}_i\frac{\partial V}{\partial x_i}\right\rangle_{E,\alpha}=\left\langle r\frac{dV}{dr}\right\rangle_{E,\alpha}$$

b) For the potential $V(r)=\hbar^2\gamma/2mr^2$ the virial theorem states that

$$2\left\langle\hat{T}\right\rangle_{E,\alpha}=\left\langle r\frac{dV}{dr}\right\rangle_{E,\alpha}=\left\langle\frac{\hbar^2\gamma}{2m}r\frac{d\left(r^{-2}\right)}{dr}\right\rangle_{E,\alpha}=-\left\langle 2\frac{\hbar^2\gamma}{2mr^2}\right\rangle_{E,\alpha}=-2\left\langle V\right\rangle_{E,\alpha}$$

It means that the average energy, which is under the assumption equal to E vanishes: $\left\langle T+V\right\rangle_{E,\alpha}=0$. Therefore there can be no bound states with negative energy!

c) If $V(r)=\hbar^2\gamma/2mr^{2-\eta}$ then the units of γ are (length)$^{-\eta}$. It means that in contrast with the previous case there is a characteristic length: $d=|\gamma|^{-1/\eta}$. By involving the mass and the Planck's constant we can construct the characteristic energy: $E_*=\frac{\hbar^2}{2md^2}=\frac{\hbar^2}{2m}|\gamma|^{1/\eta}$. It is safe to assume that there can be a bound state with the energy of the order of $-E_*$.

d) In the limit $\eta\to 0$ the characteristic length $d=|\gamma|^{-1/\eta}$ can tend either to zero or to infinity:

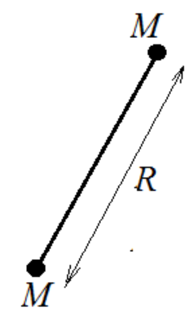
$$d\xrightarrow{\eta\to 0}=\begin{matrix}0 & |\gamma|>1\\ \infty & |\gamma|<1\end{matrix}$$

Therefore if $|\gamma|<1$ all states at $\eta=0$ are unbound and have non-negative energies, while if $|\gamma|>1$ the characteristic length vanishes, characteristic energy diverges, which means that the particle "falls" at the center: there are infinitely many states below any level!

N. Christ December 2, 2012

Quals Problem

1. Two point particles of mass M and charges $\pm Q$ are joined by a massless rod of total length R which is free to move in three dimensions inside a box of side D. Assume the wave function obeys periodic boundary conditions at the walls of the box.



   (a) Find the energy eigenvalues of this quantum system.

   (b) Find the four lowest normalized energy eigenstates of the system which have zero spatial momentum.

   (c) If a uniform electric field $\vec{E}$ is applied to the system find the shift in the ground state energy to first and second order in E.

   (d) Is the energy of the ground state increased or decreased by the presence of the electric field?

Suggested Solution

1. (a) The system separates into independent motion of the center of mass and rotation about the center of mass. If $\vec{P}$ is the center of mass momentum and $\vec{L}$ the angular momentum about the center of mass, we can write the quantum Hamiltonian as

$$H = \frac{\vec{P}^{\,2}}{4M} + \frac{\vec{L}^{\,2}}{MR^2}. \tag{1}$$

Since the operator $\vec{P}$ has eigenvalues $\frac{2\pi\hbar}{D}(n_1, n_2, n_3)$ while $\vec{L}^{\,2}$ has eigenvalues $l(l+1)\hbar^2$ where the n_i and l are integers, the energy eigenvalues are:

$$E_{\vec{n},l} = \left(\frac{2\pi\hbar}{D}\right)^2 \frac{n_1^2 + n_2^2 + n_3^2}{4M} + \frac{l(l+1)\hbar^2}{MR^2} \tag{2}$$

(b) The zero momentum wave function is independent of the cm position and will carry the normalization factor $1/D^{3/2}$. Thus, the four normalized wavefunctions are:

$$\psi(\vec{r},\theta,\phi)_{0,0} = \frac{1}{D^{3/2}}\sqrt{\frac{1}{4\pi}} \tag{3}$$

$$\psi(\vec{r},\theta,\phi)_{1,0} = \frac{1}{D^{3/2}}\sqrt{\frac{3}{4\pi}}\cos(\theta) \tag{4}$$

$$\psi(\vec{r},\theta,\phi)_{1,+1} = \frac{1}{D^{3/2}}\sqrt{\frac{3}{8\pi}}\sin(\theta)e^{i\phi} \tag{5}$$

$$\psi(\vec{r},\theta,\phi)_{1,-1} = \frac{1}{D^{3/2}}\sqrt{\frac{3}{8\pi}}\sin(\theta)e^{i\phi} \tag{6}$$

(c) There is no first order energy shift and the second order shift gets a single contribution from the $l = 1, m = 0$ state:

$$E_{0,0}^{(2)} = \sum_n \frac{|\langle n|QRE\cos(\theta)|\vec{P} = \vec{0}; 0, 0\rangle|^2}{E_{0,0}^{(0)} - E_n^{(0)}} \tag{7}$$

$$= -\frac{3R^4Q^2E^2M}{2\hbar^2} \tag{8}$$

(d) The ground state energy is lowered.

QM Problem (Heinz)

Consider a two-level quantum system described by Hamiltonian H, with states $|a>$ and $|b>$ and energies $E_a = 0$ and $E_b = E_0$. The system is initially in state $|a>$. Suppose that a constant perturbation H' is applied from time $t = 0$ until some arbitrary subsequent time t.

Find the probability $P_b(t)$ of being in state $|b>$ just after the perturbation has been removed. Sketch the variation with time t, indicating characteristic time scales on the plot. (Partial credit will be given for an accurate plot, even if the information is incomplete.) The problem should be solved *without approximation* for two forms of the perturbation H'.

(a) $<a|H'|a> = U_a$, $<b|H'|b> = U_b$, $<a|H'|b> = <b|H'|a> = 0$

(b) $<a|H'|a> = 0$, $<b|H'|b> = 0$, $<a|H'|b> = <b|H'|a> = U$.

QM Problem (Heinz)

(a) For a diagonal perturbation, the system remains in the same state $\Rightarrow P_b(t) = 0$ ||

(b) For $H = H_0 + H' = \begin{pmatrix} 0 & 0 \\ 0 & E_0 \end{pmatrix} + \begin{pmatrix} 0 & u \\ u & 0 \end{pmatrix} = \begin{pmatrix} 0 & u \\ u & E_0 \end{pmatrix}$

we have new eigenstates and eigenvalues

$$E_{A,B} = \tfrac{1}{2}\left(E_0 \mp \sqrt{E_0^2 + 4u^2}\right) \equiv \tfrac{1}{2}(E_0 \mp E')$$

$$\begin{pmatrix} |A\rangle \\ |B\rangle \end{pmatrix} = \begin{pmatrix} u/\sqrt{E_A^2+u^2} & E_A/\sqrt{E_A^2+u^2} \\ u/\sqrt{E_B^2+u^2} & E_B/\sqrt{E_B^2+u^2} \end{pmatrix} \begin{pmatrix} |a\rangle \\ |b\rangle \end{pmatrix}$$

The evolution of State $|a\rangle$ from $t=0$ under H is

$$|\psi(t)\rangle = e^{-iE_At/\hbar}|A\rangle\langle A|a\rangle + e^{-iE_Bt/\hbar}|B\rangle\langle B|a\rangle, \text{ so}$$

$$P_b(t) = |\langle b|\psi(t)\rangle|^2 = |\langle b|A\rangle\langle A|a\rangle e^{-iE_At/\hbar} + \langle b|B\rangle\langle B|a\rangle e^{-iE_Bt/\hbar}|^2$$

$$= |\langle b|A\rangle\langle A|a\rangle|^2 \, |e^{iE't/2\hbar} - e^{-iE't/2\hbar}|^2$$

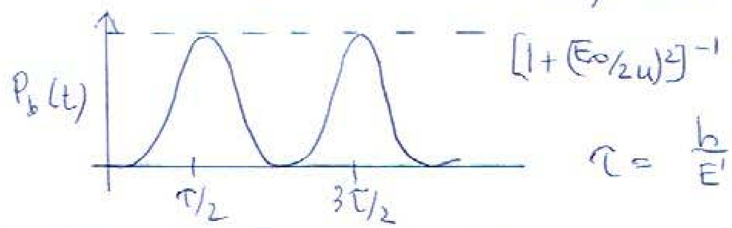
Since $P_b(0) = 0 \Rightarrow \langle b|A\rangle\langle A|a\rangle = -\langle b|B\rangle\langle B|a\rangle$ and the common phase factor $e^{-iE_0t/2\hbar}$ can be eliminated.

$$\therefore \quad P_b(t) = 4|\langle b|A\rangle\langle A|a\rangle|^2 \sin^2(E't/2\hbar)$$

$$4|\langle b|A\rangle\langle A|a\rangle|^2 = 4E_A u^2/(E_A^2+u^2)^2 \text{ from } (*)$$

$$= 1/[1+(E_0/2u)^2] \text{ after some algebra}$$

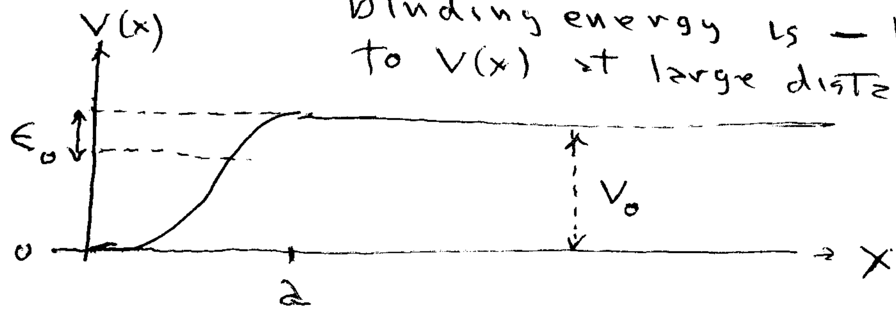
$$\| \; P_b(t) = \left[1+\left(\tfrac{E_0}{2u}\right)^2\right]^{-1} \sin^2(E't/2\hbar) \quad \text{with } E' = (E_0^2+4u^2)^{1/2}$$



$$\tau = \frac{h}{E'} = \frac{h}{\sqrt{E_0^2+4u^2}}$$

Note: The main features of this result should be apparent without detailed analysis, just from the concept of mode beating. Knowledge of the new eigenvalues yields the full plot of $P_b(t)$ except for the coefficient.

(Points out of 15) A one dimensional attractive potential well, $V(x)$, binds a mass m particle to a reflecting wall. The binding energy is $-|\epsilon_0|$ relative to $V(x)$ at large distances away.



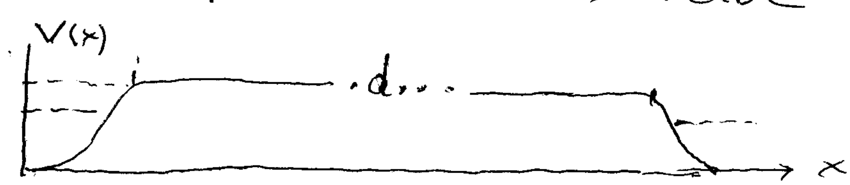
$$V(x) = V_0 \quad \text{for} \quad x > a$$
$$= 0 \quad \text{for} \quad x = 0$$

For $x \gtrsim a$ the particle wave function
$$\psi_0 = K e^{-\alpha x}$$

(3) a) What is α?

(1) b) Estimate K if the probability for the particle being inside and outside the potential well are comparable.

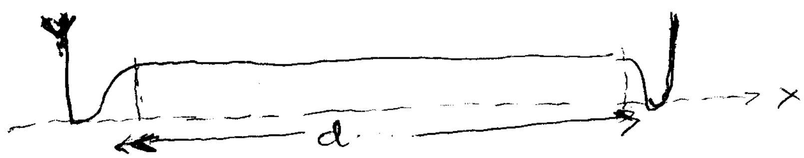
Now suppose that after a long interval d ($d \gg a$) $V(x)$ drops back to its value at $x = 0$



If at time $t = 0$ the particle is in the state with wavefunction ψ_0 the probability for still finding it near the well at later times is diminishing function of time $P(t) \sim e^{-t/\tau}$.

(3) c) Estimate τ

A reflecting wall is now also inserted at a distance $x = a + d + a$ from the first one at $x = 0$ so that the potential and reflector at either end mirror each other



(2) d) What is the new ground state from combining $\Psi_0(x)$ and $\Psi_0(d-x)$?

(3) e) Estimate its tiny energy shift, δE, relative to $-|\epsilon_0|$. [Assume that $\Psi_0 = k e^{-\alpha x}$ is an adequately approximation for all x.]

(3) f) What is the new $P(t)$?

How does time for it to drop to, say, 1/2 compare to that for c)?

Answer

a) $\frac{(2m|\epsilon_0|)^{1/2}}{\hbar}$

b) $K = e^{\alpha d} / \alpha^{1/2}$ $\left(\int_d^\infty |\psi_0|^2 dx = \frac{1}{2} \right)$

c) $\left(\frac{V_0}{\hbar}\right) e^{-2\alpha d}$

or $|\epsilon_0| / \hbar$

or $\hbar / m a^2$

The main point is the $e^{-2\alpha d}$

In our problem
$a \sim 1/\alpha$
$\langle KE \rangle \sim \langle P.E \rangle / 2$

$\epsilon_0 \sim V_0 / 2$

so any dimension analysis "guess" gives same answer

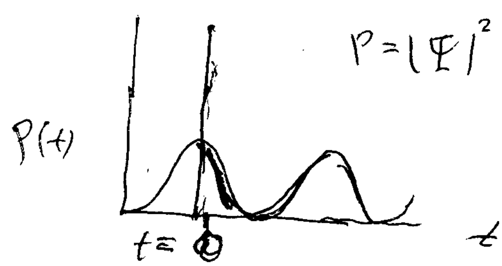
d) $\frac{\Psi_0(x) + \Psi_0(d-x)}{\sqrt{2}} \equiv \psi_{0s}$

e) $\delta E = V_0 e^{-\alpha d}$ ← $\int_0^a V(x)\, \psi_0(x)\, \psi_0(d-x)$, $\int_0^a \frac{V_0}{a} e^{-d\alpha} dx \, dx$

$\Psi = \psi_{0s} e^{-i\delta E t/\hbar} + \psi_{Es} e^{\delta E t/\hbar}$

$\psi_{0s} = \frac{\psi_0(x) - \psi_0(d-x)}{\sqrt{2}}$

f) $P = |\Psi|^2 \propto \psi_0^2 \cos^2 \delta E t/\hbar + \psi_0^2(d-x) \sin^2 \delta E t/\hbar$



$P(t)_{x \lesssim a} = \cos^2 t/\tau$ $\frac{1}{\tau} = \frac{\delta E}{\hbar}$

Ratio of times $\tau \sim e^{-\alpha d}$ for c) and $\hbar / \delta E$

1) Two spin-1/2 particles are known to be in the singlet state. Let $S_a^{(1)}$ be the component of the spin angular momentum of particle 1 in the direction defined by the unit vector $\hat{\mathbf{a}}$, and $S_b^{(2)}$ the component of the spin angular momentum of particle 2 in the direction defined by the unitvector $\hat{\mathbf{b}}$. Determine the expectation value of $S_a^{(1)} S_b^{(2)}$ as a function of the angle θ between $\hat{\mathbf{a}}$ and $\hat{\mathbf{b}}$.

1) Two spin-1/2 particles are known to be in the singlet state. Let $S_a^{(1)}$ be the component of the spin angular momentum of particle 1 in the direction defined by the unit vector ${\bf \hat a}$, and $S_b^{(2)}$ the component of the spin angular momentum of particle 2 in the direction defined by the unitvector ${\bf \hat b}$. Determine the expectation value of $S_a^{(1)S_b^{(2)}$ as a function of the angle θ between ${\bf \hat a}$ and ${\bf \hat b}$.

1) Choose $\hat{a}$ to be along the z-axis

$$\Rightarrow \quad S_a^{(1)} = S_z^{(1)}$$

$$S_b^{(2)} = \cos\theta \, S_z^{(2)} + \sin\theta \left(\cos\phi \, S_x^{(2)} + \sin\phi \, S_y^{(2)} \right)$$

For a singlet state,

$$|\psi\rangle = \frac{1}{\sqrt{2}} \left(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle \right)$$

Using $\langle \uparrow\downarrow | S_z^{(1)} | \downarrow\uparrow \rangle = 0$, we find

$$\langle \psi | S_a^{(1)} S_b^{(2)} | \psi \rangle = \frac{1}{2} \left\{ \langle \uparrow\downarrow | S_z^{(1)} S_b^{(2)} | \uparrow\downarrow \rangle + \langle \downarrow\uparrow | S_z^{(1)} S_b^{(2)} | \downarrow\uparrow \rangle \right\}$$

$$= \frac{1}{2} \left\{ \left(\frac{\hbar}{2}\right)\left(-\frac{\hbar}{2}\cos\theta\right) + \left(-\frac{\hbar}{2}\right)\left(\frac{\hbar}{2}\cos\theta\right) \right\}$$

$$= -\frac{\hbar^2}{4}\cos\theta$$