Columbia University
Department of Physics
QUALIFYING EXAMINATION
Monday, January 12, 2009
3:10 PM - 5:10 PM

Classical Physics
Section 2. Electricity, Magnetism & Electrodynamics

Two hours are permitted for the completion of this section of the examination. Choose 4 problems out of the 5 included in this section. (You will not earn extra credit by doing an additional problem). Apportion your time carefully.

Use separate answer booklet(s) for each question. Clearly mark on the answer booklet(s) which question you are answering (e.g., Section 2 (Electricity etc.), Question 2; Section 2 (Electricity etc.) Question 4, etc.)

Do **NOT** write your name on your answer booklets. Instead clearly indicate your **Exam Letter Code**.

You may refer to the single handwritten note sheet on 8 ½ x 11" paper (double-sided) you have prepared on Classical Physics. The note sheet cannot leave the exam room once the exam has begun. This note sheet must be handed in at the end of today's exam. Please include your Exam Letter Code on your note sheet. No other extraneous papers or books are permitted.

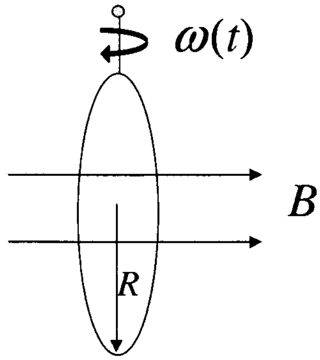
Simple calculators are permitted. However, the use of calculators for storing and/or recovering formulae or constants is NOT permitted.

Questions should be directed to the proctor.

Good luck!!

1. Calculate the spin frequency decay time, τ, of a thin ring of mass M and radius R that hangs on a string and spins with an angular frequency $\omega(t)$ in a horizontal magnetic field B. The ring has conductivity σ, and a small cross-sectional area $\pi r^2 \ll \pi R^2$.

Assume initially $\omega(0) = \omega_0$ and that the energy lost to Joule heating per period is small compared to the rotation kinetic energy at all times. You can assume the string does not exert any torque. (Hint: use $\langle \sin^2\theta(t) \rangle = ½$ over a period.)



2. An optically active medium can rotate the plane of polarization of light. The susceptibility tensor of such a medium can be expressed as:

$$\overleftrightarrow{\chi} = \begin{pmatrix} \chi_{11} & i\chi_{12} & 0 \\ -i\chi_{12} & \chi_{11} & 0 \\ 0 & 0 & \chi_{33} \end{pmatrix}$$

where $\overleftrightarrow{\chi}$ is related to the polarizability tensor in the usual fashion, $\vec{P} = \varepsilon_0 \overleftrightarrow{\chi} \cdot \vec{E}$. In $\overleftrightarrow{\chi}$, χ_{11}, χ_{12} and χ_{33} are all real. Assume a plane wave propagates in this medium in the z-direction (which is also the 3-direction) with frequency ω. Use Maxwell's equations to establish the following.

(a) That in an optically active medium the propagating EM wave is transverse.

(b) Show the medium admits EM waves with two distinct k-vectors of magnitude k_R, k_L. Find k_R, k_L in terms of ω and the necessary χ_{ij}.

(c) Show that the two k-vectors k_R, k_L correspond to the propagation of right and left circularly polarized EM waves.

(d) Find an expression for the rotary power $\equiv n_R - n_L$ in terms of the χ_{ij}.

3. Part (a) of the figure shows two coils with self-inductances L_1 and L_2. In the relative position shown, their mutual inductance is M. The positive current direction and the positive electromotive force direction in each coil are defined by the arrows in the figure. The equations relating currents and electromotive forces are

$$\mathcal{E}_1 = -L_1\frac{dI_1}{dt} \pm M\frac{dI_2}{dt} \tag{1}$$

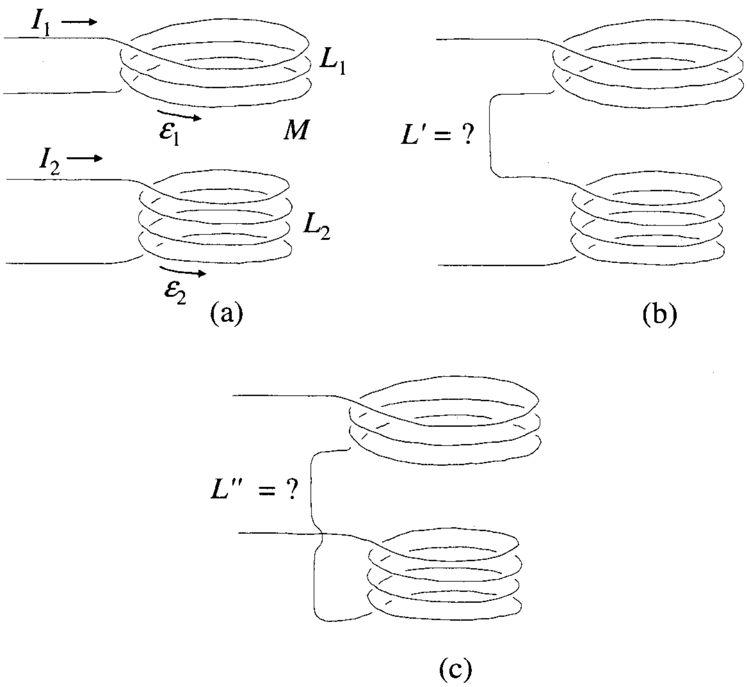
and

$$\mathcal{E}_2 = -L_2\frac{dI_2}{dt} \pm M\frac{dI_1}{dt} \tag{2}$$

Given that M is always to be taken as a positive constant, how must the signs be chosen in these equations? What if we had chosen, as we might have, the other direction for positive current, and for positive electromotive force, in the lower coil?

Now connect the two coils together as in part (b) of the figure to form a single circuit. What is the inductance L' of this circuit, expressed in terms of L_1, L_2 and M? What is the inductance L'' of the circuit formed by connecting the coils as shown in (c)? Which circuit, (b) or (c), has the greater self-inductance?

Considering that the self-inductance of any circuit must be a positive quantity, see if you can draw a general conclusion, valid for any conceivable pair of coils, concerning the relative magnitude of L_1, L_2, and M.



4. The N-th multipole moment of a charge distribution $\rho(\vec{x})$ is a rank-N tensor $M_{(N)}^{i_1 i_2 \ldots i_N}$ defined as

$$M_{(N)}^{i_1 i_2 \ldots i_N} \equiv \int d^3x \, \rho(\vec{x}) \, x^{i_1} x^{i_2} \ldots x^{i_N} \tag{1}$$

So, for instance, the monopole moment ($N = 0$) is a scalar – the total charge:

$$M_{(0)} = \int d^3x \, \rho(\vec{x}) \equiv Q \tag{2}$$

and the dipole moment ($N = 1$) is the familiar vector

$$M_{(1)}^{i} = \int d^3x \, \rho(\vec{x}) x^i \equiv p^i \tag{3}$$

(a) In equation (1) the position vector $\vec{x}$ is measured with respect to a predetermined, but arbitrary origin. Show that for a given N, the resulting value for $M_{(N)}^{i_1 i_2 \ldots i_N}$ does not depend on the choice of origin *if and only if* the lower order multipoles (those with smaller N's) vanish.

(b) Explain why this ambiguity does not impair the multipole expansion for the electrostatic potential.

5. A thin, non-conducting disk of radius R is spinning around its symmetry axis with angular velocity ω. The disk is uniformly charged with a charge density per unit area of σ.

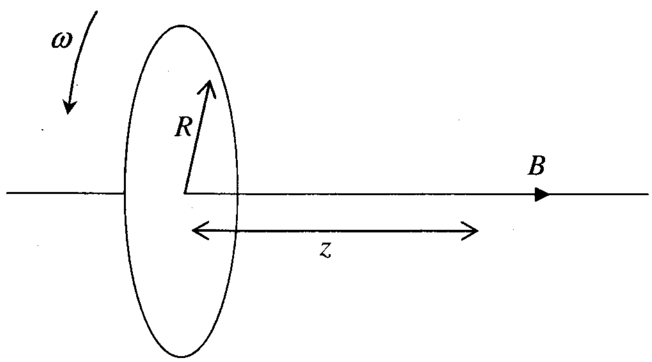
(a) What is the exact expression for the magnetic field along the symmetry axis of the disk as a function of the distance, z, from the disk?

(A useful integral may be $\int \frac{r^3 dr}{(r^2+z^2)^{3/2}} = \left(\frac{r^2+2z^2}{\sqrt{r^2+z^2}}\right))$

(b) For distances far from the disk, the disk looks like a magnetic dipole. What is the effective magnetic dipole moment?

(c) Show that the expressions in part (a) and (b) agree at large distances.

(The American physicist Henry Rowland in 1868 used such a rotating disk to show that the magnetic field due to moving charge distributions is identical with the magnetic field due to an electric current having the same geometry.)



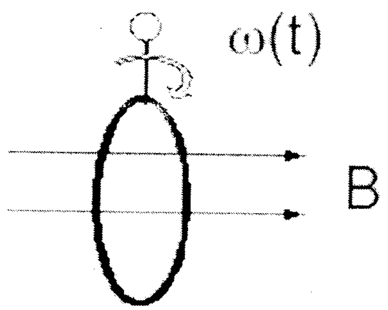
SOLUTIONS E+M

Gyulassy - E+M
Sec 2 E+M
Problem # 1

Quals 2009: Electromagnetism (M.Gyulassy)

1. Calculate the spin frequency decay time, τ, of a thin ring of mass M and radius R that hangs on a string and spins with an angular frequency $\omega(t)$ in a horizontal magnetic field B. The ring has conductivity, σ and a small area $\pi r^2 \ll \pi R^2$.

   Assume initially $\omega(0) = \omega_0$ and that the energy lost to Joule heating per period is small compared to the rotation kinetic energy at all times. (Hint: use $\langle \sin^2 \theta(t) \rangle = \frac{1}{2}$ over a period.)



Solution

The rotational kinetic energy is $KE(t) = \frac{1}{2} I_0 \omega^2$ where $I_0 = \frac{1}{2} M R^2$ for the ring. When the ring is at an angle $\theta(t)$ with respect to the horizontal constant B field, there is a magnetic flux, $\Phi(t) = B\pi R^2 \cos\theta(t)$ through the loop with $d\theta/dt = \omega(t)$. Faraday's law says that the induced EMF $E(2\pi R) = I(t)\Omega = -1/c \; d\Phi/dt = \pi B R^2 \omega(t)/c \; \sin\theta(t)$, where the ring resistance is $\Omega = (2\pi R)/(\sigma \pi r^2)$.

The induced alternating current $I(t)$ dissipates energy according to Joule heating at a rate $P = IV = \Omega I(t)^2 = (d\Phi/dt)^2/\Omega = (\pi B R^2 \omega(t)/c \; \sin\theta(t))^2 (\sigma\pi r^2)/(2\pi R)$. Over a period, the time averaged power dissipated using $< sin^2 >= \frac{1}{2}$ is $< P >= \frac{1}{2}(\pi B R^2 \omega(t)/c)^2 (\sigma \pi r^2)/(2\pi R)$.

The rotational kinetic energy decreases according to $dKE/dt = -P$. Using $M = \rho\pi r^2 (2\pi R)$ in terms of the mass density ρ, $\frac{1}{2}\,\frac{1}{2}\rho\pi r^2 (2\pi R) R^2 (2\omega\dot{\omega}) = -(\pi B R^2 \omega(t)/c)^2 (\sigma\pi r^2)/(2\pi R)$.

Therefore , $\dot{\omega} = -\omega/\tau$ where the spin relaxation time is $\tau = 4\rho c^2/(B^2\sigma)$.

Check dimensions: $[rhoc^2] = En/Vol$, $[B^2] = En/Vol$ whereas $[\sigma] = 1/Time$.

Sec 2 E+M Problem # 2 Hailey

solution

1.) Maxwell $-\frac{\partial^2 \vec{E}}{\partial z^2} + \frac{1}{c^2}\frac{\partial^2 \vec{E}}{\partial t^2} = \mu_0 \frac{\partial^2 \vec{P}}{\partial t^2}$ (1-d)

(get from $\vec{\nabla}\times\vec{E} = -\mu_0 \partial \vec{H}/\partial t$; $\vec{\nabla}\times\vec{H} = \varepsilon_0 \frac{\partial \vec{E}}{\partial t} + \frac{\partial \vec{P}}{\partial t}$ if necessary)

Put in plane wave solution $e^{i(kz-\omega t)}$

And $\overleftrightarrow{\chi}\cdot\vec{E}$ ⇒

x: $-k^2 E_x + \omega^2/c^2\, E_x = -\frac{\omega^2}{c^2}(\chi_{11} E_x + i\chi_{12} E_y)$

y: $-k^2 E_y + \omega^2/c^2\, E_y = -\omega^2/c^2(-i\chi_{12} E_x + \chi_{11} E_y)$

z: $\omega^2/c^2\, E_z = -\omega^2/c^2\, \chi_{33} E_z$

Last equation ⇒ $E_z = 0$ ⇒ transverse wave Ans

2.) The x and y equation have non-trivial solution if

$$\begin{vmatrix} -k^2 + \omega^2/c^2(1+\chi_{11}) & i\frac{\omega^2}{c^2}\chi_{12} \\ -i\,\omega^2/c^2\,\chi_{12} & -k^2 + \frac{\omega^2}{c^2}(1+\chi_{11}) \end{vmatrix} = 0$$

⇒ $k_{R,L} = \omega/c\sqrt{1+\chi_{11} \pm \chi_{12}}$ Ans

3.) sub k_R, k_L into x or y above and get

$E_x = \pm i E_y$ ⇒ circularly polarized E-M Ans

4.) ~~$n\omega = kc$~~ → $n_R - n_L = \sqrt{1+\chi_{11}+\chi_{12}} - \sqrt{1+\chi_{11}-\chi_{12}}$

Ans $n_R - n_L \propto \chi_{12}$ for $\chi_{12} \ll 1+\chi_{11}$

Sec 2 E & M
Problem # 3
Mowhinney

Solution

1. Since in part (a) of the figure, both coils have the same conventions for current flow and electromotive force, and since they are aligned so that increasing the magnetic flux through one increases the flux through the other, minus signs must be chosen for the mutual inductance term.

$$\mathcal{E}_1 = -L_1\frac{dI_1}{dt} - M\frac{dI_2}{dt} \tag{3}$$

and

$$\mathcal{E}_2 = -L_2\frac{dI_2}{dt} - M\frac{dI_1}{dt} \tag{4}$$

2. If the convention for I_2 and $\mathcal{E}_2$ is reversed, then the mutual inductance enters with a plus sign.

$$\mathcal{E}_1 = -L_1\frac{dI_1}{dt} + M\frac{dI_2}{dt} \tag{5}$$

and

$$\mathcal{E}_2 = -L_2\frac{dI_2}{dt} + M\frac{dI_1}{dt} \tag{6}$$

3. For part (b) of the figure, we have

$$\begin{aligned}\mathcal{E} &= \mathcal{E}_1 + \mathcal{E}_2 && (7)\\ &= -(L_1 + L_2 + M + M)\frac{dI}{dt} && (8)\end{aligned}$$

giving $L' = L_1 + L_2 + 2M$

4. For part (c) of the figure, we have

$$\begin{aligned}\mathcal{E} &= \mathcal{E}_1 + \mathcal{E}_2 && (9)\\ &= -(L_1 - M + L_2 - M)\frac{dI}{dt} && (10)\end{aligned}$$

giving $L'' = L_1 + L_2 - 2M$

5. Circuit (b) has the greater self-inductance

6. To keep the self-inductance of the combined circuit positive, we need $L_1 + L_2 - 2M > 0$, or $M \leq \sqrt{L_1 L_2}$. This follows since $L_1 + L_2 \pm 2\sqrt{L_1 L_2} = (\sqrt{L_1} \pm \sqrt{L_2})^2 \geq 0$.

Nicolis E&M
Sec 2 E+M
Problem #4
Nicolis

2 E&M

The N-th multipole moment of a charge distribution $\rho(\vec{x})$ is a rank-N tensor $M_{(N)}^{i_1 i_2 \cdots i_N}$ defined as

$$M_{(N)}^{i_1 i_2 \cdots i_N} \equiv \int d^3x\, \rho(\vec{x})\, x^{i_1} x^{i_2} \cdots x^{i_N} \,. \tag{19}$$

So, for instance, the monopole moment ($N = 0$) is a scalar—the total charge

$$M_{(0)} = \int d^3x\, \rho(\vec{x}) \equiv Q \,, \tag{20}$$

and the dipole moment ($N = 1$) is the familiar vector

$$M_{(1)}^{i} = \int d^3x\, \rho(\vec{x})\, x^i \equiv p^i \,. \tag{21}$$

1. In eq. (??) the position vector $\vec{x}$ is measured with respect to a predetermined, but arbitrary origin. Show that for a given N, the resulting value for $M_{(N)}^{i_1 i_2 \cdots i_N}$ does not depend on the choice of origin *if and only if* the lower order multipoles (those with smaller N's) vanish.
2. Explain why this ambiguity does not impair the multipole expansion for the electrostatic potential.

Solution

1. Suppose we choose a different origin O', displaced by $\vec{a}$ with respect to our original choice O. Then our new definition of the multipole moments is

$$M'^{\,i_1 i_2 \cdots i_N}_{(N)} \equiv \int d^3x\, \rho(\vec{x})\, (\vec{x} + \vec{a})^{i_1} (\vec{x} + \vec{a})^{i_2} \cdots (\vec{x} + \vec{a})^{i_N} \,. \tag{22}$$

For instance, for the new dipole moment we have

$$\vec{p}\,' \equiv \int d^3x\, \rho(\vec{x})\, (\vec{x} + \vec{a}) = \vec{p} + Q\, \vec{a} \,, \tag{23}$$

which indeed is equal to the original dipole moment $\vec{p}$ if and only if the monopole moment vanishes. It is clear from eq. (??) that the same conclusion holds for all N's. Indeed, by expanding the products in eq. (??) and collecting terms homogeneous in $\vec{a}$ we get schematically

$$M'^{\,i_1 i_2 \cdots i_N}_{(N)} = M_{(N)}^{i_1 i_2 \cdots i_N} + (M_{(N-1)}\, a)^{i_1 i_2 \cdots i_N} + (M_{(N-2)}\, a\, a)^{i_1 i_2 \cdots i_N} + \cdots + (M_{(0)}\, a \cdots a)^{i_1 i_2 \cdots i_N} \,, \tag{24}$$

where each term denotes a suitable tensor combination (actually, totally symmetric) of the quantities in parentheses. It is clear that the new multipole moment coincides with the original one for a generic displacement $\vec{a}$ if and only if all the lower order multipole moments vanish.

2. The multipole expansion can be thought of as an expansion in powers of (d/r), where d is the typical size of the charge distribution, and r is the distance from the charge distribution to the observation point $\vec{x}_{\rm obs}$. Of course r so defined is ambiguous, at the level of $\delta r \sim d$, for it is not specified which point inside the charge distribution we are computing the distance from. This is exactly equivalent to the ambiguity discussed above for the multipole moments. In other words, by changing our choice of origin, we change both the multipole moments and r, in such a way that the potential at $\vec{x}_{\rm obs}$ computed via the multipole expansion is unaltered. The bottom line is that, as long as we compute r *with respect to the same origin* that we use for computing the multipole moments, the multipole expansion is consistent, and origin-independent.

Sec # 2 E&M
Problem # 5

Quals Problem 1

M. Shaevitz - E+M
Fall, 2008

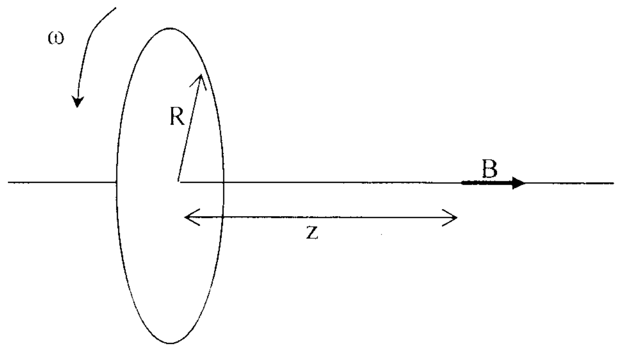
A thin, non-conducting disk of radius R is spinning around its symmetry axis with angular velocity ω. The disk is uniformly charged with a charge density per unit area of σ.

a) What is the exact expression for the magnetic field along the symmetry axis of the disk as a function of the distance, z, from the disk?
(A useful integral may be $\int \frac{r^3 dr}{\left(r^2+z^2\right)^{3/2}} = \left(\frac{r^2+2z^2}{\sqrt{r^2+z^2}}\right)$)

b) For distances far from the disk, the disk looks like a magnetic dipole. What is the effective magnetic dipole moment?

c) Show that the expressions in part a) and b) agree at large distances.

(The American physicist Henry Rowland in 1868 used such a rotating disk to show that the magnetic field due to moving charge distributions is identical with the magnetic field due to an electric current having the same geometry.)



Solution:
The current density of the disk is $di = \frac{\sigma ds dr}{dt} \Rightarrow \frac{di}{dr} = \sigma\frac{ds}{dt} = \sigma \mathrm{v} = \sigma\omega r$

a) The parallel component of the magnetic field of a current loop along the axis can be derived from the Biot-Savart $d\vec{B} = \frac{\mu_0}{4\pi}\frac{i\, d\vec{s}\times\vec{r}}{r^3}$ formula:

$$dB_{\parallel} = \int_0^{2\pi r} \frac{\mu_0}{4\pi}\frac{di}{\left(r^2+z^2\right)}\frac{r}{\sqrt{r^2+z^2}}\,dl = \frac{\mu_0 r^2 di}{2\left(r^2+z^2\right)^{3/2}}$$

One can then integrate over the disk to obtain the total field

$$B = \int_0^R \frac{\mu_0 r^2 di}{2\left(r^2+z^2\right)^{3/2}} = \int_0^R \frac{\mu_0 r^2\left(r\omega\sigma dr\right)}{2\left(r^2+z^2\right)^{3/2}}$$

$$= \frac{\mu_0\omega\sigma}{2}\int_0^R \frac{r^3 dr}{\left(r^2+z^2\right)^{3/2}} = \frac{\mu_0\omega\sigma}{2}\left(\frac{r^2+2z^2}{\sqrt{r^2+z^2}}\right)_0^R = \frac{\mu_0\omega\sigma}{2}\left(\frac{R^2+2z^2}{\sqrt{R^2+z^2}} - 2z\right)$$

b) Calculate the magnetic moment from $i \times Area$

$$\mu = \int_0^R \pi r^2 di = \int_0^R \pi r^2\left(r\omega\sigma\right)dr = \frac{\pi\omega\sigma R^4}{4}$$

c)

Calculating B_{axis} for $z \gg R$ gives $B_{axis} = \frac{\mu_0 \omega \sigma}{2}\left(\left(R^2+2z^2\right)\left(R^2+z^2\right)^{-1/2}-2z\right)$

$$= \frac{\mu_0 \omega \sigma}{2}\left(\left(R^2+2z^2\right)\frac{1}{z}\left(1-\frac{1}{2}\frac{R^2}{z^2}+\frac{3}{8}\frac{R^4}{z^4}\cdots\right)-2z\right) = \frac{\mu_0 \omega \sigma}{2}\left(\frac{R^4}{4z^3}\right)$$

For a dipole $B_{axis} = \frac{\mu_0}{2\pi}\frac{\mu_{dipole}}{z^3} \Rightarrow \mu_{dipole} = \frac{\pi \omega \sigma R^4}{4}$ which agrees with part b)