Columbia University
Department of Physics
QUALIFYING EXAMINATION
Monday, January 14, 2008
11:10 AM – 1:10 PM

Classical Physics
Section 2. Electricity, Magnetism & Electrodynamics

Two hours are permitted for the completion of this section of the examination. Choose 4 problems out of the 5 included in this section. (You will not earn extra credit by doing an additional problem). Apportion your time carefully.

Use separate answer booklet(s) for each question. Clearly mark on the answer booklet(s) which question you are answering (e.g., Section 2 (Electricity etc.), Question 2; Section 2(Electricity etc.) Question 4, etc.)

Do **NOT** write your name on your answer booklets. Instead clearly indicate your **Exam Letter Code**.

You may refer to the single handwritten note sheet on 8 ½ x 11" paper (double-sided) you have prepared on Classical Physics. The note sheet cannot leave the exam room once the exam has begun. This note sheet must be handed in at the end of today's exam. Please include your Exam Letter Code on your note sheet. No other extraneous papers or books are permitted.

Simple calculators are permitted. However, the use of calculators for storing and/or recovering formulae or constants is NOT permitted.

Questions should be directed to the proctor.

Good luck!!

1. A charge Q is uniformly distributed inside a sphere of radius R. The sphere rotates with angular velocity Ω. Find the magnitude of the magnetic field at its poles.

2. A semi-infinite material extends throughout the region $z > 0$ and has a flat boundary at $z = 0$. The region $z < 0$ is vacuum. The non-conducting ($\sigma = 0$) and non-permeable ($\mu = 1$) material has real index of refraction N_+ for circularly polarized waves with positive helicity and N_- for circularly polarized waves with negative helicity, when these waves propagate in the z-direction. $N_+ > 1$ and $N_- > 1$, but $N_+ \neq N_-$. There is no free charge and no free current on the surface of the material or within the volume of the material.

A plane wave linearly polarized in the x-direction and propagating in the z-direction in the vacuum region is incident normally on the material's flat surface. The reflected and transmitted waves also propagate along the z-direction. Calculate the ratio of the reflected intensity to the incident intensity, and describe the polarization of the reflected wave.

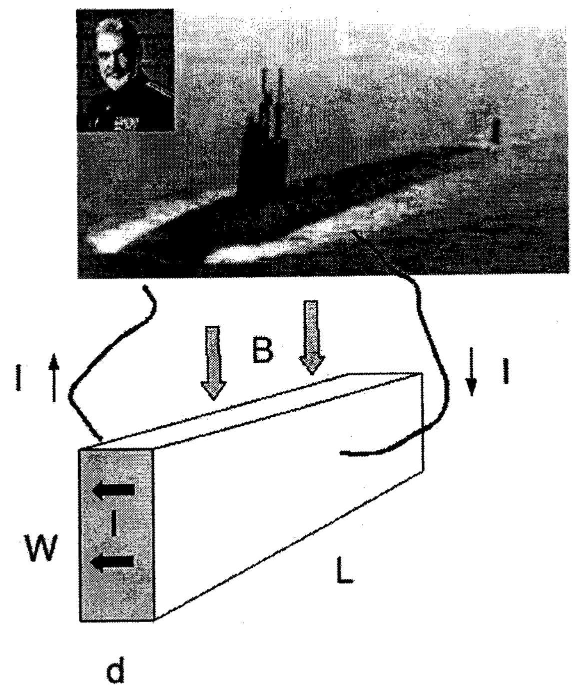
3. A laser beam, constant in time, is shot into space. Show that far from the Earth the transverse dimension of the beam scales as the square root of the wavelength. To be concrete you may consider a beam propagating in the z direction with a Gaussian profile in the transverse directions:

$$\vec{E} = \vec{E}_0 e^{i(kz-\omega t)} \psi(x, y, z)$$

where $\vec{E}_0$ is a polarization vector and

$$\psi(x, y, z) = \exp\left(A(z) - \frac{x^2 + y^2}{2B(z)} \right)$$

4. In a popular film Admiral Connery stole a Russian nuclear submarine with a secret silent magnetohydrodynamic MHD drive. Consider two metal plates with length L = 100 m, width $W = 10$ m separated by a distance $d = 5$ m as shown in the figure. Assume that a P = 100 Megawatt nuclear reactor is available to power DC generators on board. DC currents between the plates involve the mobile Na and Cl ions in the sea.



(a) Suppose a voltage difference V is applied to the drive. Given the sea water conductivity $\sigma = 5/(\Omega\text{–m})$ what is the total electric resistance R in Ohms of the drive? What is the current I in Amps if V = 1 Volt is applied? What is the theoretical limit of I that the reactor could deliver given its power, P, if there were no other sources of resistance or losses?

(b) Compute the total thrust force, F, in Newtons of a modest MHD drive assuming I = 1000 A and assuming the salt ions only feel the earth's magnetic field $B \approx 0.5$ Gauss.

(c) Compute the thrust if an external uniform 10 Tesla B field is also turned on directed toward the Earth over the whole drive. Compare this increased thrust to the conventional propeller thrust $F_{\text{prop}} = 10^7$ Newtons that the reactor powers to move it at the maximum cruising speed of 20 Knots. How does the drive thrust depend on dimensions d, L and W?

5. A perpendicularly incident beam of right circularly polarized light is reflected by a stationary mirror. Show that the reflected beam is left circularly polarized.

Problem: Charge Q is uniformly distributed inside a sphere of radius R. The sphere rotates with angular velocity Ω. Find the magnitude of magnetic field at its poles.

Solution:

The rotating charge produces current density $\vec{j} = \rho \vec{v}_{\rm rot}$, where $\rho = 3Q/4\pi R^3$. One can view the rotating sphere as a collection of rings of radii a carrying currents

$$dI = j\, dS = \rho\, \Omega\, a\, dS,$$

where dS is an element of vertical cross section of the sphere. Magnetic field $dB_{\rm pole}$ created by current loop dI is

$$dB_{\rm pole} = \frac{2\pi\, a^2\, dI}{c\, b^3},$$

where b is the distance between the loop and the pole. Integrating over dI (half cross section of the sphere), one finds ($\sin\alpha \equiv a/b$)

$$B_{\rm pole} = \frac{2\pi\,\rho\,\Omega}{c}\int \frac{a^3}{b^3}\, dS = \frac{2\pi\,\rho\,\Omega}{c}\int\!\!\int \sin^3\alpha\; b\, d\alpha\, db = \frac{2\pi\,\rho\,\Omega}{c}\int_0^{2R} db\, b \int_{b/2R}^{1} d\cos\alpha\, \sin^2\alpha = \frac{2}{5}\,\frac{Q\Omega}{cR}.$$

[The magnetic dipole moment of the sphere is $\mu = (1/5)R^2\Omega\, Q/c$. One can show that the sphere produces dipole magnetic field at all $r \geq R$, so $B_{\rm pole} = 2\mu/R^3$.]

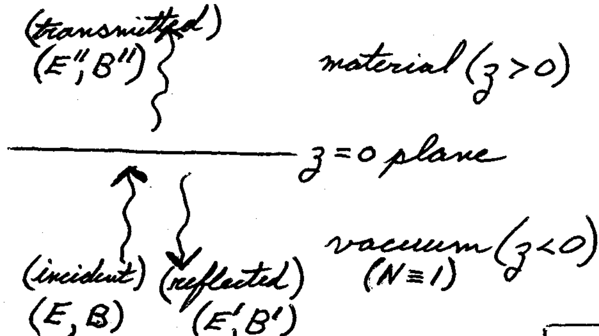
Section 2, E & M, # 2
"Faraday rotator"
Identified by Miklos 12/14/07

ELECTROMAGNETISM PROBLEM
ALLAN BLAER

A semi-infinite material extends throughout the region $z > 0$ and has a flat boundary at $z = 0$. The region $z < 0$ is vacuum. The non-conducting ($\sigma = 0$) and non-permeable ($\mu = 1$) material has real index of refraction N_+ for circularly polarized waves with positive helicity and real index of refraction N_- for circularly polarized waves with negative helicity, when these waves propagate in the z-direction. $N_+ > 1$ and $N_- > 1$, but $N_+ \neq N_-$. There is no free charge and no free current on the surface of the material or within the volume of the material.

A plane wave linearly polarized in the x-direction and propagating in the z-direction in the vacuum region is incident normally on the material's flat surface. The reflected and transmitted waves also propagate along the z-direction. Calculate the ratio of the reflected intensity to the incident intensity, and describe the polarization of the reflected wave.

Section 2 (E&M) Problem 2 - Allan Blaer



$\vec{E}_{inc} = E\,\hat{x}\,e^{i(kz-\omega t)}$

where $\hat{x} = \frac{1}{2}\left[(\hat{x}+i\hat{y}) + (\hat{x}-i\hat{y})\right]$.

This expresses the linear polarization as a superposition of ⊕ and ⊖ helicity circular polarizations.

Let $\hat{e}_\pm \equiv \frac{1}{\sqrt{2}}(\hat{x} \pm i\hat{y}) \Longrightarrow \boxed{\vec{E}_{inc} = \frac{E}{\sqrt{2}}(\hat{e}_+ + \hat{e}_-)e^{i(kz-\omega t)}}$.

Consider the reflection and transmission of each helicity state separately.

In general for plane waves with $\vec{k} = \hat{n}k$ and $N \equiv \frac{kc}{\omega}$:

$$\vec{B} = N\hat{n}\times\vec{E} \quad \left(\text{from } \operatorname{curl}\vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t}\right).$$

Therefore, $\vec{E}_\pm = E_0\,\hat{e}_\pm\,e^{i(kz-\omega t)} \Rightarrow \vec{B}_\pm = N\hat{z}\times\vec{E}_\pm = \mp iN\vec{E}_\pm$ (incident, transmitted)

and $\vec{E}_\pm = E_0\,\hat{e}_\pm\,e^{i(-kz-\omega t)} \Rightarrow \vec{B}_\pm = -N\hat{z}\times\vec{E}_\pm = \pm iN\vec{E}_\pm$ (reflected)

At $z=0$, $\vec{E}_{tan}$ is continuous $\Longrightarrow \boxed{E_\pm + E_\pm' = E_\pm''}$.

At $z=0$, $\vec{H}_{tan}$ is continuous $\overset{(\mu=1)}{\Longrightarrow}$ $\vec{B}_{tan}$ is continuous $\Longrightarrow \mp iE_\pm \pm iE_\pm' = \mp iN_\pm E_\pm''$

$$\therefore \boxed{E_\pm - E_\pm' = N_\pm E_\pm''}.$$

Solving simultaneously $\Longrightarrow \boxed{E_\pm' = \frac{(1-N_\pm)}{(1+N_\pm)}E_\pm}$ and $\boxed{E_\pm'' = \left(\frac{2}{1+N_\pm}\right)E_\pm}$

Therefore, $\vec{E}_{inc} = \frac{E}{\sqrt{2}}(\hat{e}_+ + \hat{e}_-)e^{i(kz-\omega t)}$ implies that

$$\boxed{\vec{E}_{reflected} = \frac{E}{\sqrt{2}}\left[\frac{(1-N_+)}{(1+N_+)}\hat{e}_+ + \frac{(1-N_-)}{(1+N_-)}\hat{e}_-\right]e^{i(-kz-\omega t)}}$$

Ratio of time-averaged intensities $= R = \frac{I_{reflected}}{I_{incident}} = \frac{\vec{E}^*_{reflected}\cdot\vec{E}_{reflected}}{\vec{E}^*_{inc}\cdot\vec{E}_{inc}}$

$$\therefore \boxed{R = \frac{\left[\left(\frac{1-N_+}{1+N_+}\right)^2 + \left(\frac{1-N_-}{1+N_-}\right)^2\right]}{2}}$$

(no cross terms between ⊕ and ⊖ helicities are present.)

Rewriting $\vec{E}_{reflected}$ in terms of $\hat{x}$ and $\hat{y}$ $\Longrightarrow$ elliptical polarization

$$\vec{E}_{reflected} = \frac{E}{2}\left\{\hat{x}\left[\frac{(1-N_+)}{(1+N_+)} + \frac{(1-N_-)}{(1+N_-)}\right] + i\hat{y}\left[\frac{(1-N_+)}{(1+N_+)} - \frac{(1-N_-)}{(1+N_-)}\right]\right\}e^{i(-kz-\omega t)}$$

Section 2, E & M, # 3
"Wave spreading"
Identified by Miklos 12/14/07

Brooijmans

Quals 08, Optics

December 9, 2007

Problem

A laser beam, constant in time, is shot to the moon. Show that the transverse dimension of the beam scales as the square root of the wavelength. As an example, take a beam with a gaussian profile:

$$\psi_0(\overrightarrow{r}) = exp\left(A(z) - \frac{r^2}{2B(z)}\right), \tag{1}$$

where ψ_0 is the amplitude of one of the wave components at $z = 0$, z is the direction of motion and $r^2 = x^2 + y^2$.

Solution

The beam propagates according to the wave equation

$$\nabla^2 \overrightarrow{E} - \frac{1}{c^2}\frac{\partial^2 \overrightarrow{E}}{\partial^2 t} = 0. \tag{2}$$

While we don't know how the beam is polarized, it's clear that each component, ψ, propagates independently according to the wave equation. In general, we can write

$$\psi(\overrightarrow{r}, t) = \psi_0(\overrightarrow{r}, t)e^{i(kz-\omega t)}. \tag{3}$$

Here $\omega = kc$. Substituting and neglecting second derivatives of ψ_0 with respect to z (since the beam changes little along the direction of motion compared to the transverse direction), we get

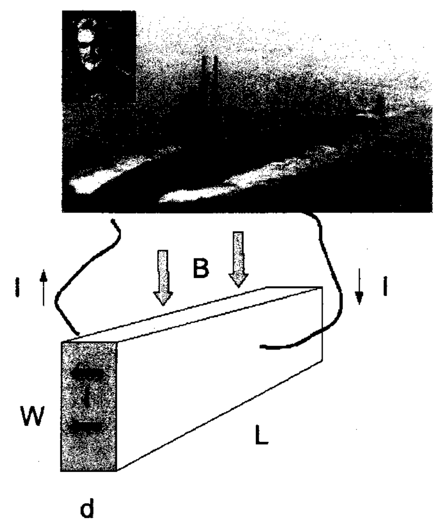
$$\frac{\partial \psi_0}{\partial z} = \frac{i}{2k}\left[\frac{\partial^2 \psi_0}{\partial x^2} + \frac{\partial^2 \psi_0}{\partial y^2}\right], \tag{4}$$

QED. Injecting our example function we find

$$\frac{\partial A}{\partial z} = \frac{-i}{kB}, \frac{\partial B}{\partial z} = \frac{i}{k}. \tag{5}$$

Qualifying exam
EM 2
M. Gyulassy

2 In a popular film Admiral Connery stole a Russian nuclear submarine with a secret silent magnetohydrodynamic MHD drive. Consider two metal plates with length $L = 100$ m, width $W = 10$ m seperated by a distance $d = 5$ m as shown in the figure. Assume that a $P = 100$ Megawatt nuclear reactor is available to power DC generators on board. DC currents between the plates involve the mobile Na and Cl ions in the sea.



a) Suppose a voltage difference V is applied to the drive. Given the sea water conductivity $\sigma = 5/(Ohm - m)$ what is the total electric resistance R in Ohms of the drive? What is the current I in Amps if $V = 1$ Volt is applied? What is the theoretical limit of I that the reactor could deliver given its power, P, if there were no other sources of resistance or losses?

b) Compute the total thrust force, F, in Newtons of a modest MHD drive assuming $I = 1000$ A and assuming the salt ions only feel the earth's magnetic field $B \approx 0.5$ Gauss.

c) Compute the thrust if an external uniform 10 Tesla B field is also turned on directed toward the earth over the whole drive. Compare this increased thrust to the conventional properler thrust $F_{prop} = 10^7$ Newtons that the reactor powers to move it at the maximum cruising speed of 20 Knots. How does the drive thrust depend on dimensions d, L and W?

Solution Gyulassy EM2:

a) Ohms Law $J = \sigma E = \sigma V/d$. Total current is $I = J \times LW = V/R$ so that the drive electric resistance is $R = d/(LW\sigma) = 5m(0.2\Omega - m)/10^3 m^2 = 10^{-3}\Omega$. For $V = 1$ Volt the total current across the drive is then $I = 1,000$ Amps. In theory $P = IV = I^2R$ can power a lot higher current in the absence of other losses so that $I_{max} = \sqrt{10^8 10^3} \approx 10^6/\sqrt{10}$ Amps.

b) For N ions within the drive volume the current density is $J = Nqv/(dLW) = I/(LW)$, so that $Nqv = Id$. The total magnetic force is then $F = NqvB = IdB = \sigma V(LW)B$ that is independent of d if V is fixed, but indep of L, W if I is fixed.

For the budget MHD drive that uses only the earth magnetic field and a 1000 A rated generator, $F = IdB = (1000A)5m(0.5 \times 10^{-4} Tesla) = 0.25$ Newtons, which is way too small to move the sub.

The total reactor power $P = 10^8$ W can support a max speed of $v_{max} = 20$ Knots $= 10$ m/s under conventional noisy propeller propulsion. The maximum conventional thrust is thus $F_{max} = P/v_{max} = 10^8 W/10m/s = 10^7$ Newtons.

c) If we used advanced superconductor technology to create a 10 Tesla uniform field over the entire 5000 m^3 volume to make a superduper MHD drive, then we can still can only achieve $F = 0.25N \times 10T/5 \times 10^{-5}T = 5 \times 10^4$ Newtons which is still 200 times smaller than what conventional propeller technology can deliver. With drive segmentation and dimension tuning and using multiple DC generators higher thrusts could be obtained.

Sec 2 #5
Marta

Assuming perpendicular incidence onto a conductive layer mirror the, Fresnel formulae tell us that:

$$E_R = \left(\frac{1-\beta}{1+\beta}\right)E_I = rE_I$$

where the amplitude reflectivity coefficient '*r*' is approaching unity for perfect conductors ($\sigma \to \infty$). They also tell us that upon reflection the reflected field experiences a 180° phase shift relatively to the incident field.

$$E_R = -E_I$$

A right circularly polarized monochromatic plane wave propagating along the *z* axis can be represented as

$$\bar{E}^{RIGHT} = \bar{E}_x + \bar{E}_y \qquad \bar{E}_x = E_0 \cos(kz - wt)\hat{x}$$
$$\bar{E}_y = E_0 \sin(kz - wt)\hat{y}$$

while the left circularly polarized wave can be represented as

$$\bar{E}^{LEFT} = \bar{E}_x - \bar{E}_y$$

Please note the 180° phase difference between the arguments of the sin and cos terms in contrast to the right circularly polarized wave.

The general validity of the discussion is not compromised by the zero phase angle at t=0 at z=0. We may also choose to place the mirror at z=0 in the x-y plane and choose the wave vector ***k*** of the incident right circularly polarized wave positive (propagating from $-\infty \to z=0$).

Upon reflection, <u>both</u> the ***x*** and the ***y*** component of the ***E*** fields acquire a 180° phase shift, so in itself it would not lead to a differential 180° phase shift between the two components required when changing from right to left circular polarization. Besides this common mode phase shift, the wave vector ***k*** also becomes negative (propagating from $z=0 \to -\infty$). Since the nature of the circularly polarized light is described from the viewpoint of propagation direction, this sign change of ***k*** will change the chirality.