Columbia University
Department of Physics
QUALIFYING EXAMINATION

Wednesday, January 15, 2014
3:10PM to 5:10PM
Modern Physics
Section 4. Relativity and Applied Quantum Mechanics

Two hours are permitted for the completion of this section of the examination. Choose 4 problems out of the 5 included in this section. (You will not earn extra credit by doing an additional problem). Apportion your time carefully.

Use separate answer booklet(s) for each question. Clearly mark on the answer booklet(s) which question you are answering (e.g., Section 4 (Relativity and Applied QM), Question 2, etc.).

Do **NOT** write your name on your answer booklets. Instead, clearly indicate your **Exam Letter Code**.

You may refer to the single handwritten note sheet on $8\frac{1}{2}$" $\times$ 11" paper (double-sided) you have prepared on Modern Physics. The note sheet cannot leave the exam room once the exam has begun. This note sheet must be handed in at the end of today's exam. Please include your Exam Letter Code on your note sheet. No other extraneous papers or books are permitted.

Simple calculators are permitted. However, the use of calculators for storing and/or recovering formulae or constants is NOT permitted.

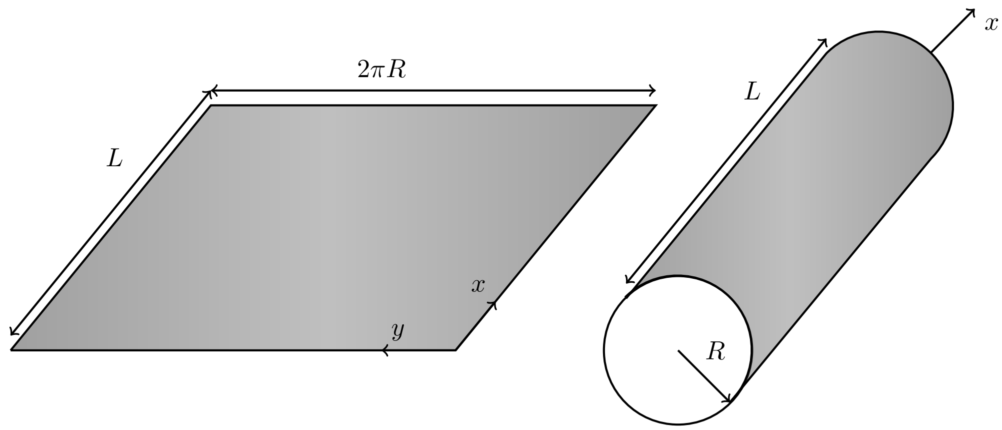
Questions should be directed to the proctor.

Good Luck!

1. An ideal plane mirror moves with velocity $\vec{v}$ normal to its surface. A photon incident at angle θ (with respect to the normal) with energy E is reflected by the mirror. Find the photon angle θ_{new} and energy E_{new} after reflection.

2. Consider a free electron confined in a 2-dimensional stripe whose length is L and width is $2\pi R$ ($L \gg R$). The Hamiltonian of the system is then expressed by $\frac{\hbar^2}{2m}\left(\left(\frac{\partial}{\partial x}\right)^2 + \left(\frac{\partial}{\partial y}\right)^2\right)$, where x and y are along the long and short edge of the stripe.

   We then roll this stripe up to form a cylindrical shell (see figure below). Assume the cylindrical axis is along x direction.



   (a) Using the coordinate x and y defined on the surface of the cylinder, write down the wave function using a suitable set of good quantum numbers. We assume that L is very large and thus momentum along x direction is a good quantum number

   (b) Express the energy of electrons in terms of the quantum numbers you find in (a).

   (c) Find the density of states as a function of energy in the limit of very large L.

3. Consider operation of the Large Hadron Collider (LHC), colliding protons with a center-of-mass energy of 8 TeV (=8000 GeV). Consider the process $pp \to Z' + X$, where Z' is a new heavy boson of mass 2 TeV. Assume the Z' is produced at rest in the lab frame, and always decays to a top-quark and an anti-top quark (ie. $Z' \to t\bar{t}$).

   (a) Calculate the momentum of the top quark in the lab frame.

   (b) Assume the top quark immediately decays to a b-quark and a W boson (ie. $t \to Wb$). The ability to separate the jet due to the b-quark from the decay products of the W boson depends on their separation in the detector; for angular separations (measured in radians) which are smaller than the typical jet size of 0.4, the decay products will start to overlap in the detector.

   Determine the angular separation between the W boson and b-quark from the top quark decay, considering the (not so typical) case where they are produced with equal momentum in the lab frame.

   (Assume the following mass values: m(top) = 175 GeV, m(W) = 80 GeV, m(b) = 5 GeV.)

4. Consider an observer following a space-time trajectory $x^\mu(\tau)$, where τ stands for proper time. Define the observer's acceleration four-vector as

$$a^\mu \equiv \frac{d^2 x^\mu}{d\tau^2}$$

(a) Assuming that the observer only moves along one spatial direction— say x— compute the $x^\mu(\tau)$ corresponding to constant positive acceleration α, in the sense that

$$a^\mu a_\mu = \alpha^2 = \text{const}, \qquad a^x > 0. \tag{1}$$

Plot such a trajectory in a Minkowski diagram. [*Hint: argue that the Lorentz invariance of the condition (1) motivates a simple ansatz for the trajectory, and solve for the parameters of that ansatz.*]

(b) In your solution for $x^\mu(\tau)$, call $(\tilde{t}, \tilde{x})$ the space-time point at which the observer is instantaneously at rest (in the coordinates that you are using). Suppose that a light signal gets emitted at $(\tilde{t}, \tilde{x} - 2/\alpha)$, towards the observer. Will the observer ever receive the signal? Draw in the same Minkowski diagram as above the unobservable region, that is, the space-time region from which the observer will never be able to receive signals.

[Notation: $c = 1$; $\eta_{\mu\nu}$ = diag(-,+,+,+).]

5. A spin-less particle of charge $-e$ and mass m is constrained to move in the x-y plane. There is a constant magnetic field B along the direction normal to the plane. Assume that the field derives from a vector potential that has a single component along the x-direction given by $A_x = -By$.

   (a) Write the expression for the Hamiltonian of one particle.

   (b) To find the solutions of the Schrödinger equation for the stationary states consider wavefunctions

   $$\psi(x, y) = f(x)\phi(y)$$

   where

   $$f(x) = \exp\left(\frac{i}{\hbar}p_x x\right)$$

   and p_x is the x-component of momentum. Write the Schrödinger equation for $\phi(y)$ and obtain the expression for the spectrum of energy levels E_n (Landau levels) in the field B. What are the quantum numbers that correspond to a Landau level?

   (c) Assume that the area of the plane is the product: $L_x L_y$, the lengths L_x and L_y along the x- and y-directions respectively. Also assume that the function $f(x)$ satisfies the 'obvious' boundary condition

   $$f(x = 0) = f(x = L_x)$$

   Find the degeneracy of a Landau level as function of magnetic field for $L_x = L_y = L$.

Relativity:
An ideal plane mirror moves with velocity $\vec{v}$ normal to its surface. A photon incident at angle θ (with respect to the normal) with energy E is reflected by the mirror. Find the photon angle $\theta_{\rm new}$ and energy $E_{\rm new}$ after reflection.

Solution:
Chose the x-axis along the normal to the mirror, $\vec{n}$, and the x-y plane so that it contains the photon velocity and $\vec{n}$. The photon 4-momentum before reflection is $c^{-1}(E, E\cos\theta, E\sin\theta, 0)$. The 4-momentum components in the rest-frame of the mirror are found using Lorentz boost along the x-axis. The t- and x-components of this transformation give

$$E' = \gamma(1+\beta\cos\theta)E, \quad E'\cos\theta' = \gamma(\cos\theta+\beta)E \qquad \Rightarrow \qquad \cos\theta' = \frac{\cos\theta+\beta}{1+\beta\cos\theta},$$

where $\beta = v/c$ and $\gamma = (1-\beta^2)^{-1/2}$. In this frame, the photon reflection is described by

$$E'_{\rm new} = E', \qquad \cos\theta'_{\rm new} = -\cos\theta'.$$

The photon angle and energy in the lab frame are obtained by the inverse transformation ($\beta \to -\beta$),

$$\cos\theta_{\rm new} = \frac{\cos\theta'_{\rm new} - \beta}{1-\beta\cos\theta'_{\rm new}} = \frac{-\cos\theta' - \beta}{1+\beta\cos\theta'} = -\frac{(1+\beta^2)\cos\theta + 2\beta}{1+\beta^2+2\beta\cos\theta},$$

$$E_{\rm new} = \gamma(1-\beta\cos\theta'_{\rm new})E'_{\rm new} = \gamma(1+\beta\cos\theta')E' = \gamma^2(1+\beta^2+2\beta\cos\theta)E.$$

Applied Quantum Solutions

(a) The free particle like wave function can be written as $\Psi(x,y) = \frac{1}{\sqrt{2\pi RL}} \exp ik_x x \exp ik_y y$, where $\hbar k_x$ and $\hbar k_y$ are the momentum along x and y direction. Considering the periodic boundary condition, $k_y = n/R$, where n is an integer. Along x direction, owing to the translational invariance, momentum $p = \hbar k_x$ is a good quantum number. Summarizing,

$$\Psi_{nk}(x,y) = \frac{1}{\sqrt{2\pi RL}} \exp i \left[kx + \frac{n}{R} y \right]. \tag{1}$$

(b) Energy of the system can be given by

$$E_n(k) = \frac{\hbar^2}{2m} \left[k^2 + \left(\frac{n}{R} \right)^2 \right] \tag{2}$$

(c) The density of states can be obtained by integrating over k for the available energy E matching Eq(2): $DOS(E) = \frac{2}{2\pi} \int \delta(E - E_n(k)) dk$.

We consider the spin degeneracy 2 in the above. Here, $dE' = \frac{\hbar^2 k}{m} dk = \sqrt{\frac{2\hbar^2}{m}} \left[E' - \frac{\hbar^2}{2m} \left(\frac{n}{R} \right)^2 \right]^{1/2}$. Then, we obtain

$$DOS(E) = \frac{1}{\pi} \sum_n \int \frac{\delta(E - E') dE'}{\sqrt{\frac{2\hbar^2}{m}} \left[E' - \frac{\hbar^2}{2m} \left(\frac{n}{R} \right)^2 \right]} \tag{3}$$

Evaluating this integration, we have:

$$DOS(E) = \sum_n \sqrt{\frac{2m}{h^2}} \left[E - \frac{\hbar^2}{2m} \left(\frac{n}{R} \right)^2 \right]^{-1/2} \tag{4}$$

Relativity

Question: Consider operation of the Large Hadron Collider (LHC), colliding protons with a center-of-mass energy of 8 TeV (=8000 GeV). Consider the process pp → Z' + X, where Z' is a new heavy boson of mass 2 TeV. Assume the Z' is produced at rest in the lab frame, and always decays to a top-quark and an anti-top quark (ie. $Z' \to t\bar{t}$).

(a) Calculate the momentum of the top quark in the lab frame.
(b) Assume the top quark immediately decays to a *b*-quark and a W boson (ie. $t \to Wb$). The ability to separate the jet due to the *b*-quark from the decay products of the W boson depends on their separation in the detector; for angular separations (measured in radians) which are smaller than the typical jet size of 0.4, the decay products will start to overlap in the detector.
   Determine the angular separation between the W boson and *b*-quark from the top quark decay, considering the (not so typical) case where they are produced with equal momentum in the lab frame.

(Assume the following mass values: m(top) = 175 GeV, m(W) = 80 GeV, m(*b*) = 5 GeV.)

Answer:

(a)
Z' produced at rest, so E(Z') = m(Z').
Conservation of momentum implies top and anti-top produced with equal and opposite momenta, of magnitude p.

⇒ p(t) = p(tbar) ≡ p

Conservation of E then ⇒ E(Z') = E(t) + E(tbar) = sqrt(p^2 + mt^2) + sqrt(p^2 + mt^2)

Solve for p: p = sqrt[(m(Z')^2/4 – mt^2]

Plug in numbers to get p = 985 GeV.

(b) Let angle of each of W and b wrt top quark direction be theta (ie. 2 X theta is the angular separation between the two of them). Denote p(W) = p(b) = p.

Conservation of momentum

p(t) = 2 p costheta

Conservation of energy

E(t) = sqrt(p^2 + M^2) + sqrt(p^2 + m^2)

with M = m(W) and m = m(b).
Re-arrange (and square both sides twice) to solve for p:

p = sqrt [(E(t)^2 +M^2 – m^2)^2/(4 E(t)^2) – M^2]

Plug in numbers to get p = 497 GeV.

So, theta = cos-1 (p(t)/2p) = cos-1(985/2/497) = 0.135

Finally, angular separation is 2 theta ~ 0.27 radians (ie. smaller than jet size of 0.4).

2 Relativity: the Rindler horizon

Consider an observer following a space-time trajectory $x^\mu(\tau)$, where τ stands for proper time. Define the observer's acceleration four-vector as

$$a^\mu \equiv \frac{d^2 x^\mu}{d\tau^2} \; . \tag{14}$$

1. Assuming that the observer only moves along one spatial direction—say x—compute the $x^\mu(\tau)$ corresponding to constant positive acceleration α, in the sense that

   $$a^\mu a_\mu = \alpha^2 = \text{const} \; , \qquad a^x > 0 \; . \tag{15}$$

   Plot such a trajectory in a Minkowski diagram. [*Hint: argue that the Lorentz invariance of the condition* (15) *motivates a simple ansatz for the trajectory, and solve for the parameters of that ansatz.*]

2. In your solution for $x^\mu(\tau)$, call $(\tilde{t}, \tilde{x})$ the space-time point at which the observer is instantaneously at rest (in the coordinates that you are using). Suppose that a light signal gets emitted at $(\tilde{t}, \tilde{x} - 2/\alpha)$, towards the observer. Will the observer ever receive the signal? Draw in the same Minkowski diagram as above the unobservable region, that is, the space-time region from which the observer will never be able to receive signals.

[Notation: $c = 1$; $\eta_{\mu\nu} = \text{diag}(-, +, +, +)$.]

Solution

1. The fact that the acceleration is constant along the observer's trajectory, in the Lorentz-invariant sense (15), suggests that the trajectory $x^\mu(\tau)$ itself will describe a Lorentz-invariant curve in space-time. The only possible curves with these properties are 45° lines, and hyperbolas with asymptotes at 45°. Both these possibilities are described by the Lorentz-invariant equation

$$(x^\mu - \bar{x}^\mu)(x_\mu - \bar{x}_\mu) = R^2 \ , \tag{16}$$

where R is an arbitrary "radius" parameter—which will depend on our acceleration α—and $\bar{x}^\mu$ is an arbitrary "center", which without loss of generality we can set to the origin,

$$\bar{x}^\mu = (0, 0) \ . \tag{17}$$

The 45° lines case corresponds to $R = 0$, and describes light rays. Our accelerated observer will have instead $R^2 > 0$. The $R^2 < 0$ case in unphysical, since it describes *space-like* trajectories.

A natural ansatz for our trajectory is thus the usual parameterization of an hyperbola:

$$t = R \sinh(\beta\tau) \ , \qquad x = R \cosh(\beta\tau) \ , \tag{18}$$

which clearly obeys (16) (with $\bar{x}^\mu \to 0$.) We have to solve for the two parameters R, β. Taking the second derivatives as in (14), and plugging into (15) we get (upon using $\cosh^2 - \sinh^2 = 1$)

$$R^2 \beta^4 = \alpha^2 \ . \tag{19}$$

The second condition comes from imposing that the four-velocity

$$u^\mu \equiv \frac{dx^\mu}{d\tau} \tag{20}$$

be normalized as $u^\mu u_\mu = -1$, which implies

$$-R^2 \beta^2 = -1 \ . \tag{21}$$

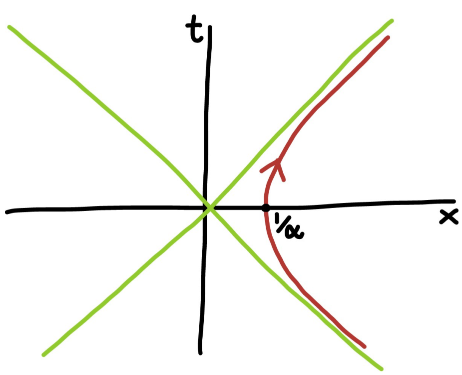
We thus get

$$R = 1/\alpha \ , \qquad \beta = \alpha \ , \tag{22}$$

so that our trajectory reads

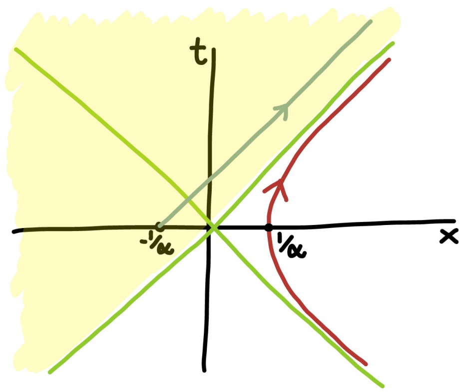
$$x^\mu = \frac{1}{\alpha}(\sinh \alpha\tau, \cosh \alpha\tau) \ . \tag{23}$$

and looks like this:



2. Our observer moves faster and faster, approaching the speed of light in the asymptotic future. It is thus not guaranteed that all light signals can reach him or her.

   The observer comes at rest at $(\tilde{t}, \tilde{x}) = (0, 1/\alpha)$. If a signal gets emitted at $(\tilde{t}, \tilde{x} - 2/\alpha) = (0, -1/\alpha)$, the observer will never receive it, as clear from the Minkowski diagram below. In fact, the observer will never receive any signal that gets emitted on the left of the $t = x$ line, which is said to be an *horizon*:



Quals 2014: Applied Quantum Mechanics

A spin-less particle of charge –e and mass m is constrained to move in the x-y plane. There is a constant magnetic field B along the direction normal to the plane. Assume that the field derives from a vector potential that has a single component along the x-direction given by $A_x = -By$.

(a) write the expression for the Hamiltonian of one particle.

(b) to find the solutions of the Schroedinger equation for the stationary states consider wavefunctions

$\psi(x,y)= f(x)\phi(y)$

where

$f(x)= \exp[(i/\hbar)p_x x]$

and p_x is the x-component of momentum.

Write the Schroedinger equation for $\phi(y)$ and obtain the expression for the spectrum of energy levels E_n (Landau levels) in the field B.
What are the quantum numbers that correspond to a Landau level?

(c) Assume that the area of the plane is the product: $L_x L_y$. The lengths L_x and L_y along the x- and y-directions respectively.
Also assume that the function f(x) satisfies the 'obvious' boundary condition

$f(x=0)=f(x=L_x)$

Find the degeneracy of a Landau level as function of magnetic field for $L_x = L_y = L$.

Quals 2014 – Problem AQM 4-5

Solution

(a) $$H = \frac{1}{2m}\left(\vec{p} - \frac{e}{c}\vec{A}\right)$$

$$\vec{A} = (A_x, 0, 0) \; ; \; A_x = -By$$

$$H = \frac{1}{2m}\left(p_x + \frac{e}{c}By\right) + \frac{1}{2m}p_y^2$$

(b) $$\frac{d^2\phi(y)}{dy^2} + \frac{2m}{\hbar^2}\left[E_n - \frac{1}{2}m\omega_c^2(y-y_0)^2\right]\phi(y) = 0$$

$$\omega_c = \frac{eB}{mc} \; ; \; |y_0| = \frac{cp_x}{eB} \; ; \; E_n = \left(n + \frac{1}{2}\right)\hbar\omega_c$$

$n = 0, 1, 2, \ldots$ and p_x are quantum numbers

(c) From the boundary condition

$$p_x = m\, 2\pi \frac{\hbar}{L} \; ; \; m \equiv \text{integer}$$

$$y_0 = \frac{2\pi c \hbar}{eBL}\, m \leq L$$

The number of states is equal to m for the value $y_0 = L \Rightarrow N = (hc/eB)^{-1} A; \; A = L^2$