Columbia University
Department of Physics
QUALIFYING EXAMINATION

Wednesday, January 10, 2018
2:00PM to 4:00PM
Modern Physics
Section 4. Relativity and Applied Quantum Mechanics

Two hours are permitted for the completion of this section of the examination. Choose 4 problems out of the 5 included in this section. (You will not earn extra credit by doing an additional problem). Apportion your time carefully.

Use separate answer booklet(s) for each question. Clearly mark on the answer booklet(s) which question you are answering (e.g., Section 4 (Relativity and Applied QM), Question 2, etc.).

Do **NOT** write your name on your answer booklets. Instead, clearly indicate your **Exam Letter Code**.

You may refer to the single handwritten note sheet on $8\frac{1}{2}$" $\times$ 11" paper (double-sided) you have prepared on Modern Physics. The note sheet cannot leave the exam room once the exam has begun. This note sheet must be handed in at the end of today's exam. Please include your Exam Letter Code on your note sheet. No other extraneous papers or books are permitted.

Simple calculators are permitted. However, the use of calculators for storing and/or recovering formulae or constants is NOT permitted.

Questions should be directed to the proctor.

Good Luck!

1. Solve both parts of this problem.

   (a) A π^- particle stops in Deuterium and is captured from the atomic ground state creating a pair of outgoing neutrons. Examine the possible combinations of spin and orbital angular momentum of the pair of neutrons and explain why the existence of this process implies that the parity of the π^- must be negative. (Recall that the Deuteron has spin one.)

   (b) A π^- with total energy E_π strikes a hydrogen target producing a Λ^0 and a K^0:

$$\pi^- + P \to \Lambda^0 + K^0.$$

   What is the minimum energy E_π that will allow this reation to take place? (Recall $M_{\pi^-} = 140$ MeV, $M_{\text{proton}} = 938$ MeV, $M_{\Lambda^0} = 1116$ MeV and $M_{K^0} = 498$ MeV.)

2. A train rushes by a platform moving to the right at speed $v = 12c/13$. Call the midpoint of the train and the midpoint of the platform the "origin" of each frame, and note that as the respective origins pass one another, clocks at those locations both read 4 PM. As the train's conductor, who is sitting in the engine room (at the front of the train), passes an observer on the platform, there's an explosion in the conductor's control console.The platform observer notes that when the explosion happens, a clock in the control room reads 2 PM.

   (a) According to those on the train, what is the reading on the clock at the origin of platform when the explosion takes place?

   (b) According to those on the train, how far is the origin of the platform from the origin of the train frame when the explosioni takes place?

   (c) The condcutor immediately sends a distress signal (traveling at light speed) alerting his colleagues of the explosion. At what time does his colleague at the origin of the train receive the message?

   (d) When the observer at the origin of the train receives the signal, how far away will he/she claim he/she is from the origin of the platform?

3. The velocity v of a particle does not transform particularly nicely under Lorentz transformations. Consider however the rapidity η, defined as

$$\eta = \frac{1}{2} \log \frac{c+v}{c-v}$$

(a) Show that rapidity is *additive*, in the sense that if a particle is moving along x with rapidity η in the lab frame, and the lab frame is moving with respect to a second frame with rapidity η_0, also along x, then the rapidity of our particle as seen from the second reference frame is $\eta + \eta_0$.

(b) Now we want to generalize this nice property to more general setups in 3D, e.g. a particle moving along x in the lab frame, which is itself moving along y with respect to a second frame. Argue that there is no vector generalization $\vec{\eta}$ of rapidity that retains the additive property above. That is, one would like to have that under a Lorentz transformation of rapidity $\vec{\eta}_0$, the rapidity $\vec{\eta}$ of a particle transforms into $\vec{\eta}+\vec{\eta}_0$. Argue that this is impossible. [*Hint: Recall that Lorentz transformations do not commute, in the sense that when performing two subsequent Lorentz transformation along different axes, the order in which they are performed matters.*]

4. Two observers move in opposite directions in a circle of radius R with constant angular velocities ω_1 and ω_2. When they first meet, they synchronize their clocks. When they meet again, whose clock will be delayed and by how much?

5. A hydrogenic ion consists of a single electron bound to a nucleus with charge Ze, where e is the elementary charge. For a hydrogenic ion, find the scaling with Z of

   (a) the expectation value of r (the electron-nucleus separation)

   (b) the expectation value of E (the total energy of the electron)

   (c) $|\Psi(r = 0)|^2$ (the probability of finding the electron at the nucleus)

   (d) the fine-structure energy splitting

Problem:

1. Solve both parts of this problem.

   (a) A π^- particle stops in Deuterium and is captured from the atomic ground state creating a pair of outgoing neutrons. Examine the possible combinations of spin and orbital angular momentum of the pair of neutrons and explain why the existance of this process implies that the parity of the π^- must be negative. (Recall that the Deuteron has spin one.)

   (b) A π^- with total energy E_π strikes a hydrogen target producing a Λ^0 and a K^0:
   $$\pi^- + P \to \Lambda^0 + K^0.$$
   What is the minimum energy E_π that will allow this reation to take place? (Recall $M_{\pi^-} = 140$ MeV, $M_{\text{proton}} = 938$ MeV, $M_{\Lambda^0} = 1116$ MeV and $M_{K^0} = 498$ MeV.)

Solution:

(a) The initial atomic ground state has $l = 0$ and $J = 1$ and the parity of the π^-. The parity of the final state will be $(-1)^l$ where l is the orbital angular momentum of the pair of neutrons. If $l = 0$ coservation of angular momentum will requre the neutrons to be in a spin state with total spin $s = 1$. In this case both the spin and spatial wave functions are symmetric which is not possible since neutrons are Fermions. If $l = 1$ then either $S = 0$ or $S = 1$ is allowed because in both cases a combination of orbital and spin angular momentum giving a total angular momentum of one is possible. Since the $l = 1$ spatial wave function is anti-symmetric, the neutrons must also have $S = 1$ since that spin state is symmetric.

(b) For the reaction to take place the total center of mass energy must be equal or greater than $E_{\text{cm}} = M_{\Lambda^0} + M_{K^0}$. If p_{π^-} and p_{proton} are the four-momenta of initial pion and the proton target, this implies

$$(M_{\Lambda^0} + M_{K^0})^2 \le -(p_{\pi^-} + p_{\text{proton}})^2 = M_{\pi^-}^2 + M_{\text{proton}}^2 + 2E_\pi M_{\text{proton}}$$

where the right-hand side of the left-most equality has been evaluated in the laboratory system. Thus,

$$E_\pi \ge \frac{1}{2M_{\text{proton}}} \left\{ (M_{\Lambda^0} + M_{K^0})^2 - M_{\pi^-}^2 - M_{\text{proton}}^2 \right\} = 909 \text{ MeV}.$$

Problem:

1. A very long train rushes by a platform moving to the right at speed $v = 12c/13$. Call the midpoint of the train and the midpoint of the platform the "origin" of each frame, and note that as the respective origins pass one another, clocks at those locations both read 4 PM. As the train's conductor, who is sitting in the engine room (at the front of the train), passes an observer standing at the origin of the platform, there's an explosion in the conductor's control console. The observer on the platform notes that when the explosion happens, a clock in the engine room reads 2 PM.

   (a) According to those on the train, what is the reading on the clock at the origin of platform when the explosion takes place?

   (b) According to those on the train, how far is the origin of the platform from the origin of the train frame when the explosion takes place?

   (c) The conductor immediately sends a distress signal (traveling at light speed) alerting his colleagues of the explosion. At what time does his colleague at the origin of the train receive the message?

   (d) When the observer at the origin of the train receives the signal, how far away will he/she claim he/she is from the origin of the platform?

Solution:

(a) Let (x_P, t_P) and (x_T, t_T) label the postion and time coordinates of the explosion in the platform and train reference systems. These are related by the Lorentz transformation

$$x_P = \gamma(x_T + v\, t_T) \tag{1}$$
$$t_P = \gamma(t_T + v/c^2\, x_T) \tag{2}$$

where time should be measured relative to 4 PM. Thus, $t_T = -2$ hour and $x_P = 0$. This implies that $x_T = 2\,v$ hour and $t_P = t_T/\gamma = -10/13$ hour. Thus, the clock on the platform will read $4 - 10/13 = 3\frac{3}{13}$ PM.

(b) Since the time of the explosion is 2 PM on the train and the platform is moving at $v/c = -12/13$, the origin of the platform will be a distance $2\,v$ hours $= 24/13$ light hours to the right.

(c) The signal must travel a distance $x_T = 2\,v$ hours moving at light speed. This will require a time of $2\,v/c$ hours and will arrive when the clock at the origin of the train reads $2 + 2\,v/c = 2(25/13)$ or $3\frac{11}{13}$ hours.

(d) Since the time is $2/13$ hour before the origins coincide, the distance will be $24/169$ light hours.

Problem:

1. The velocity v of a particle does not transform particularly nicely under Lorentz transformations. Consider however the rapidity η, defined as

$$\eta = \frac{1}{2}\log\frac{c+v}{c-v}$$

   (a) Show that rapidity is *additive*, in the sense that if a particle is moving along x with rapidity η in the lab frame, and the lab frame is moving with respect to a second frame with rapidity η_0, also along x, then the rapidity of our particle as seen from the second reference frame is $\eta + \eta_0$.

   (b) Now we want to generalize this nice property to Lorentz boosts in three dimensions *e.g.* a particle moving along x in the lab frame, which is itself moving along y with respect to a second frame. Argue that there is no vector generalization $\vec{\eta}$ of rapidity that retains the additive property above. That is, under a Lorentz boost of rapidity $\vec{\eta}_0$, one would like the rapidity $\vec{\eta}$ of a particle to transform into $\vec{\eta}+\vec{\eta}_0$. Argue that this is impossible. [*Hint: Recall the properties of the Lorentz transformation that results when a seqence of two Lorentz boosts are performed in different directions.*]

Solution:

(a) Recall that a particle moving with velocity v in the x-direction in the lab frame will have a velocity v' in a second frame with respect to which the lab frame is moving with velocity v_0 where v' is given by:

$$v' = \frac{v+v_0}{1+\frac{vv_0}{c^2}}.$$

Thus the rapidity in this second frame is given by:

$$\eta' = \frac{1}{2}\ln\left(\frac{1+v'}{1-v'}\right) = \frac{1}{2}\ln\left(\frac{1+\frac{v+v_0}{1+\frac{vv_0}{c^2}}}{1-\frac{v+v_0}{1+\frac{vv_0}{c^2}}}\right) = \frac{1}{2}\ln\left(\frac{(1+v_0)(1+v)}{1-v_0)(1-v)}\right) = \eta_0+\eta$$

(b) We intend that the addition of two rapidity vectors $\vec{\eta}_0$ and $\vec{\eta}$ gives a third vector $\vec{\eta}_0+\vec{\eta} = \vec{\eta}+\vec{\eta}_0$. However, if two Lorentz boosts performed in different directions are combined, the resulting transformation is a combination of a boost and a rotation, not a simple boost. Thus the set of boosts does not close under composition. Further, the result depends on the order in which the two boosts are performed while vector addition commutes.

Problem:

1. Two observers move in opposite directions in a circle of radius R with constant angular velocities ω_1 and ω_2. When they first meet, they synchronize their clocks. When they meet again, whose clock will be delayed and by how much?

Solution:
$v_1 = R\omega_1$, $\Delta\tau_1 = \Delta t/\gamma_1 = \sqrt{1-v_1^2}\,\Delta t$ and $v_2 = R\omega_2$, $\Delta\tau_2 = \Delta t/\gamma_2 = \sqrt{1-v_2^2}\,\Delta t$
So if $\omega_1 < \omega_2$, then the clock of observer 1 will have seen more time pass.
$2\pi = \omega_1\Delta t + \omega_2\Delta t \rightarrow \Delta t = 2\pi/(\omega_1+\omega_2)$ So the difference on the clocks will be given by

$$\Delta\tau_1 - \Delta\tau_2 = \frac{2\pi}{\omega_1+\omega_2}\left(\sqrt{1-R^2\omega_1^2} - \sqrt{1-R^2\omega_2^2}\right). \tag{1}$$

Problem:

1. A hydrogenic ion consists of a single electron bound to a nucleus with charge Ze, where e is the elementary charge. For a hydrogenic ion, find the scaling with Z of:

   (a) The expectation value of r (the electron-nucleus separation).

   (b) The expectation value of E (the total energy of the electron).

   (c) $|\psi(r=0)|^2$ (the probability of finding the electron at the nucleus).

   (d) The fine-structure energy splitting.

Solution:

(a) The atomic radius can be obtained by balancing the Coulomb energy with the energy of quantum confinement:

$$\frac{\hbar^2}{m\langle r\rangle^2} \sim \frac{Ze^2}{\langle r\rangle}. \tag{1}$$

Therefore,

$$\langle r\rangle \propto \frac{1}{Z}. \tag{2}$$

(b) The potential energy scaling is given by

$$\langle V\rangle \sim \frac{Ze^2}{\langle r\rangle} \propto Z^2. \tag{3}$$

The same scaling applies to kinetic energy, according to the Virial theorem.

(c)

$$|\psi(r=0)|^2 \propto \frac{1}{\langle r\rangle^3} \propto Z^3. \tag{4}$$

(d)

$$\begin{aligned} \Delta_{\mathrm{FS}} &\propto \mu_e B_{\text{nucleus}} \propto B_{\text{nucleus}} \propto v_e E_{\text{nucleus}} \\ &\propto (Z\alpha c)\left(\frac{Ze}{\langle r\rangle^2}\right) \propto Z^4. \end{aligned} \tag{5}$$

Alternatively, the lowest-order relativistic kinetic energy contribution is $\langle E_{\text{kin}}^{\text{FS}}\rangle \propto p^4 \propto \langle V\rangle^2 \propto Z^4$ using the result of part (b).