# Columbia University
# Department of Physics
# QUALIFYING EXAMINATION

Monday, January 13, 2014
1:00PM to 3:00PM
Classical Physics
Section 1. Classical Mechanics

Two hours are permitted for the completion of this section of the examination. Choose 4 problems out of the 5 included in this section. (You will not earn extra credit by doing an additional problem). Apportion your time carefully.

Use separate answer booklet(s) for each question. Clearly mark on the answer booklet(s) which question you are answering (e.g., Section 1 (Classical Mechanics), Question 2, etc.).

Do **NOT** write your name on your answer booklets. Instead, clearly indicate your **Exam Letter Code**.

You may refer to the single handwritten note sheet on $8\frac{1}{2}$" × 11" paper (double-sided) you have prepared on Classical Physics. The note sheet cannot leave the exam room once the exam has begun. This note sheet must be handed in at the end of today's exam. Please include your Exam Letter Code on your note sheet. No other extraneous papers or books are permitted.

Simple calculators are permitted. However, the use of calculators for storing and/or recovering formulae or constants is NOT permitted.

Questions should be directed to the proctor.

Good Luck!

1. A solid cylinder of mass $M$ and radius $R$, rotating with angular speed $\omega_0$ about an axis through its center, is set on a horizontal surface for which the kinetic friction coefficient is $\mu_k$.

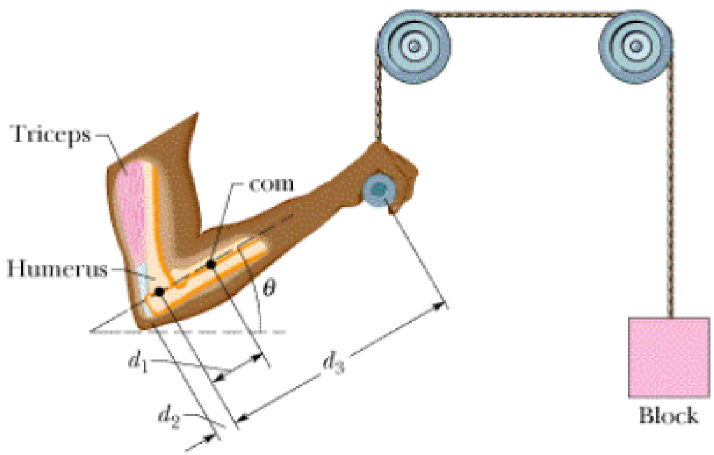
   (a) Calculate the linear acceleration of the center of mass, and the angular acceleration of rotation about the center of mass.

   (b) The cylinder is initially slipping completely. Eventually, the cylinder is rolling without slipping. Calculate the distance the cylinder moves before slipping stops.

   (c) Calculate the work done by the friction force on the cylinder as it moves from where it was set down to where it begins to roll without slipping.

   Note: the cylinder has rotational inertia $I_{cm} = \frac{1}{2}MR^2$

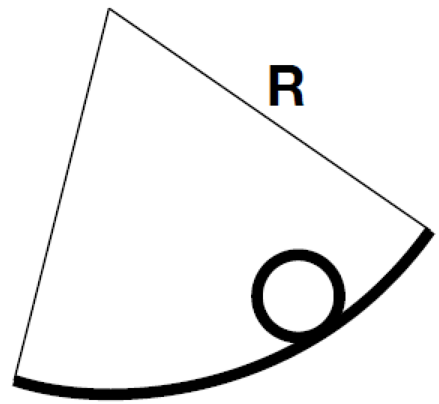
2. In the figure below, a 20 kg block is held in place via a pulley system. The person's upper arm is vertical; the forearm makes angle $\theta = 30^0$ with the horizontal. Forearm and hand together have a mass of 2 kg, with a center of mass at distance $d_1 = 10$ cm from the contact point of the forearm bone and the upper-arm bone (humerus). The triceps muscle pulls vertically on the forearm at distance $d_2 = 4.9$ cm from the contact point. Distance $d_3$ is 30 cm. What are the forces on the forearm from (a) the triceps muscle and (b) the humerus? Take $g = 9.8$ m/s$^2$.



3. A spider is hanging by a silk thread from a tree in NYC. Find the orientation and the value of the equilibrium angle that the thread makes with the vertical (i.e. with the direction of gravity), taking into account the rotation of the Earth.

   Assume that the latitude of NYC is $\theta \approx 41^0$ and the radius of the Earth is $R \approx 6,400$ km.

4. A swing of mass $m$ made from an arc section of radius of $R$ is suspended from a pivot by ropes at both ends. A hoop, also of mass $m$, and radius $a$ rolls without slipping on the swing. The swing and the hoop move without dissipative friction subject to a constant gravitational force $F_g = -mg\hat{j}$. Find all possible frequencies of oscillation of the system for small displacements from equilibrium assuming $a/R \ll 1$.



5. Consider the diatomic molecule oxygen ($O_2$) which is rotating in the $xy$ plane about the $z$ axis. The $z$ axis passes through the center of the molecule and is perpendicular to its length. At room temperature, the average separation between the two oxygen atoms is $1.21 \times 10^{-10}$m (the atoms are treated as point masses).

   (a) Calculate the moment of inertia of the molecule about the $z$ axis.

   (b) If the angular velocity of the molecule about the $z$ axis is $2.00 \times 10^{12}$ rad/s, what is its rotational kinetic energy? The molar mass of oxygen is 16 g/mol.

2014 Quals Question: Mechanics (Dodd)

A solid cylinder of mass M and radius R, rotating with angular speed $\omega_0$ about an axis through its center, is set on a horizontal surface for which the kinetic friction coefficient is $\mu_k$.

a). Calculate the linear acceleration of the center of mass, and the angular acceleration of rotation about the center of mass.

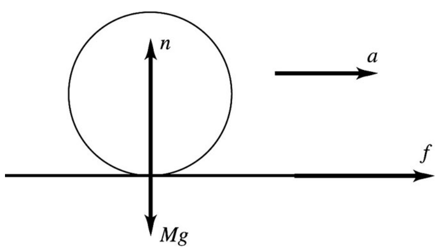
b). The cylinder is initially slipping completely. Eventually, the cylinder is rolling without slipping. Calculate the distance the cylinder moves before slipping stops.

c). Calculate the work done by the friction force on the cylinder as it moves from where it was set down to where it begins to roll without slipping.

Note: the cylinder has rotational inertia $I_{\text{cm}} = \frac{1}{2}MR^2$.

Solution:

As the cylinder is placed on the surface, we have the following forces acting:




a). The friction force is $f = \mu_k n = \mu_k Mg$, so $a = \mu_k g$.

The magnitude of the angular acceleration is $\frac{fR}{I} = \frac{\mu_k MgR}{(1/2)MR^2} = \frac{2\mu_k g}{R}$.

b). Setting $v = at = \omega R = (\omega_0 - \alpha t)R$ once the cylinder reaches the rolling condition, and solving for *t* gives

$$t = \frac{R\omega_0}{a + R\alpha} = \frac{R\omega_0}{\mu_k g + 2\mu_k g} = \frac{R\omega_0}{3\mu_k g},$$

and

$$d = \frac{1}{2}at^2 = \frac{1}{2}(\mu_k g)\left(\frac{R\omega_0}{3\mu_k g}\right)^2 = \frac{R^2\omega_0^2}{18\mu_k g}.$$

c). The final kinetic energy is $(3/4)Mv^2 = (3/4)M(at)^2$, so the change in kinetic energy is

$$\Delta K = \frac{3}{4}M\left(\mu_k g \frac{R\omega_0}{3\mu_k g}\right)^2 - \frac{1}{4}MR^2\omega_0^2 = -\frac{1}{6}MR^2\omega_0^2.$$

So the work done by the friction force is

$$W = \frac{1}{6}MR^2\omega_0^2.$$

Sec 1 - 2

# 1 Mechanics Problem

In Figure 1, a 20 kg block is held in place via a pulley system. The person's upper arm is vertical; the forearm makes angle $\theta = 30°$ with the horizontal. Forearm and hand together have a mass of 2 kg, with a center of mass at distance $d_1 = 10$ cm from the contact point of the forearm bone and the upper-arm bone (humerus). The triceps muscle pulls vertically on the forearm at distance $d_2 = 4.9$ cm from the contact point. Distance $d_3$ is 30 cm. What are the forces on the forearm from (a) the triceps muscle and (b) the humerus? Take $g = 9.8$ m/s$^2$.

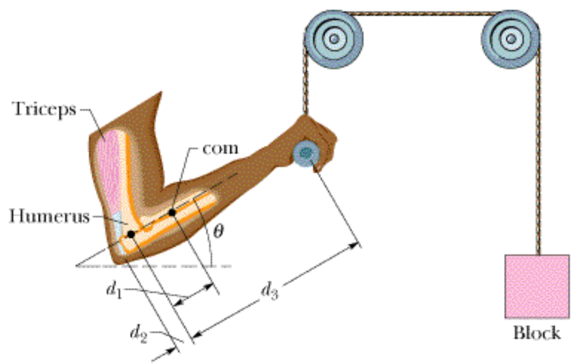



Figure 1: A person holding a weight.

## 1.1 Solution

Let's assume the triceps applies an upward force $F_t$ and the force $F_h$ from the humerus is downward. For part a, look at the torques around the pivot point:

$$\tau_{net} = 0 = (4.9cmF_t + 10cmF_g - 30cmT)\, sin30°. \quad (1)$$

This yields $F_t = 1160$N, so $F_t$ points upwards indeed. For part b, along the vertical we have

$$F_t + T - F_g - F_h = 0, \quad (2)$$

so $F_h = 1336.4$ N and $F_h$ indeed points downward.

CLASSICAL PHYSICS – MECHANICS

**A spider on a thread. SOLUTION.**

Since the spider is stationary, the thread will be deflected from the vertical due to the centrifugal force, but not the Coriolis force. The force on the spider from the thread is

$$\vec{F}_s = -\vec{F}_g - \vec{F}_c = Mg\hat{z} + M\vec{\Omega} \times (\vec{\Omega} \times \vec{R}). \quad (1)$$

Here, the $\hat{z}$ axis points vertically through the spider away from the center of the Earth, and $\vec{R} = R\hat{z}$. We also define an orthogonal $\hat{x}$ axis through the spider, such that the Earth's angular speed is

$$\vec{\Omega} = \Omega(\hat{z}\sin\theta - \hat{x}\cos\theta), \quad (2)$$

i.e. $\hat{x}$ points 90° south of $\hat{z}$. We find

$$\vec{\Omega} \times \vec{R} = \Omega R\cos\theta\hat{y} \quad (3)$$

and

$$\vec{\Omega} \times (\vec{\Omega} \times \vec{R}) = -\Omega^2 R\cos\theta(\hat{x}\sin\theta + \hat{z}\cos\theta). \quad (4)$$

Then from eq. (1) we obtain

$$\vec{F}_s = M\hat{z}(g - \Omega^2 R\cos^2\theta) - (M\Omega^2 R/2)\hat{x}\sin(2\theta). \quad (5)$$

Thus the angle is south of the vertical, and its value is

$$\alpha = \tan^{-1}\left[\frac{(\Omega^2 R/2)\sin(2\theta)}{g - \Omega^2 R\cos^2\theta}\right] \approx \tan^{-1}\left[\frac{\Omega^2 R\sin(2\theta)}{2g}\right]. \quad (6)$$

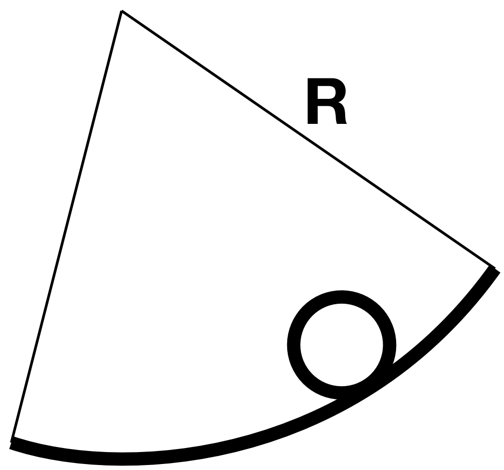
Substituting the numerical values, we obtain

$$\alpha \approx 1.7 \text{ mrad} \approx 0.1^\circ. \quad (7)$$

Here we used the rate of the Earth's rotation $\Omega = (2\pi \text{ rad})/(24 \text{ h}) \approx 7.3 \times 10^{-5}$ rad/s.

Problem

A swing of mass $m$ made from an arc section of radius of $R$ is suspended from a pivot by ropes at both ends. A hoop, also of mass $m$, and radius $a$ rolls without slipping on the swing. The swing and the hoop move without dissipative friction subject to a constant gravitational force $F_g = -mg\hat{j}$. Find all possible frequencies of oscillation of the system for small displacements from equilibrium assuming $a/R << 1$.




Solutions

Represent the motion of the system in terms of two angles that represent the angular displacements of the swing and hoop from equilibrium, $\phi$ and $\theta$, respectively. The rolling of the hoop can be described by an angular velocity $\omega_{hoop}$ which will be taken to be positive for the hoop rolling counter-clockwise.. The rolling of the hoop depends on the difference between $\dot{\phi}$ and $\dot{\theta}$ since if $\theta$ and $\phi$ increase or decrease together, the hoop remains in the same position with respect to the swing. Then, the rolling without slipping condition is $a\omega_{hoop} = R\left(\dot{\phi} - \dot{\theta}\right)$. The kinetic energy of the system is then,

$$T = \frac{1}{2}m\left(R^2\dot{\phi}^2 + (R-a)^2\dot{\theta}^2\right) + \frac{1}{2}I_{hoop}\,\omega^2, \qquad (1)$$

with $I_{hoop} = ma^2$. Substituting $a^2\omega^2 = R^2\left(\dot{\phi} - \dot{\theta}\right)^2$ and taking $R - a \to R$,

$$T = mR^2\left(\dot{\phi}^2 + \dot{\theta}^2 - \dot{\theta}\dot{\phi}\right) \tag{2}$$

The potential energy of the system can be expressed

$$U = mgR\left(1 - \cos\phi\right) + mg(R - a)\left(1 - \cos\theta\right). \tag{3}$$

Again taking $R - a \to R$ the complete Lagrangian for the system is

$$L = mR^2\left(\dot{\phi}^2 + \dot{\theta}^2 - \dot{\theta}\dot{\phi}\right) - mgR\left(2 - \cos\phi - \cos\theta\right). \tag{4}$$

Writing out the Euler-Lagrange equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial\dot{\phi}}\right) - \frac{\partial L}{\partial\phi} = mR^2\left(2\ddot{\phi} - \ddot{\theta}\right) + mgR\sin\phi = 0, \tag{5}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial\dot{\theta}}\right) - \frac{\partial L}{\partial\theta} = mR^2\left(2\ddot{\theta} - \ddot{\phi}\right) + mgR\sin\theta = 0. \tag{6}$$

Taking the small displacement limit, $\sin\theta \to \theta$ and $\sin\phi \to \phi$, we obtain coupled harmonic oscillator equations of motion,

$$2\ddot{\phi} - \ddot{\theta} = -\frac{g}{R}\phi, \tag{7}$$

$$2\ddot{\theta} - \ddot{\phi} = -\frac{g}{R}\theta. \tag{8}$$

There are many ways to solve for the frequencies of such a system. Taking a "brute force" but straight-forward way, assume normal mode solution(s) such that $\phi = c_1 e^{i\omega t}$ and $\theta = c_2 e^{i\omega t}$. For convenience, define $\gamma = \frac{g}{R\omega^2}$. Then,

$$-\omega^2\left(2c_1 - c_2\right) = -\frac{g}{R}c_1 \quad\rightarrow\quad 2c_1 - c_2 = \gamma c_1 \tag{9}$$

$$-\omega^2\left(2c_2 - c_1\right) = -\frac{g}{R}c_2 \quad\rightarrow\quad 2c_2 - c_1 = \gamma c_2 \tag{10}$$

Re-writing this as an eigenvalue equation,

$$\begin{bmatrix} 2-\gamma & -1 \\ -1 & 2-\gamma \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = 0 \tag{11}$$

We obtain the characteristic equation by requiring

$$\mathrm{Det}\begin{bmatrix} 2-\gamma & -1 \\ -1 & 2-\gamma \end{bmatrix} = 3 - 4\gamma + \gamma^2 = 0 \to \gamma = 1, 3. \tag{12}$$

So, the two possible frequencies are $\omega = \sqrt{\frac{g}{R}}$ and $\omega = \sqrt{\frac{g}{3R}}$.

# Quals Problem #1

*January 2014*

## Problem

Consider the diatomic molecule oxygen ($O_2$) which is rotating in the $xy$ plane about the $z$ axis. The $z$ axis passes through the center of the molecule and is perpendicular to its length. At room temperature, the average separation between the two oxygen atoms is $1.21 \times 10^{-10}$ m (the atoms are treated as point masses). ($a$) Calculate the moment of inertia of the molecule about the $z$ axis. ($b$) If the angular velocity of the molecule about the $z$ axis is $2.00 \times 10^{12}$ rad/s, what is its rotational kinetic energy? The molar mass of oxygen is 16 g/mol.

# Suggested Solution to Quals Problem #1

*January 2014*

## Solution

(*a*) $I = \sum m_i r_i^2 = m(d/2)^2 + m(d/2)^2 = md^2/2 = (2.66 \times 10^{-26}\ \text{kg})(1.21 \times 10^{-10}\ \text{m})^2/2 = 1.95 \times 10^{-46}\ \text{kg} \cdot \text{m}^2$
(*b*) $K = I\omega^2/2 = (1.95 \times 10^{-46}\ \text{kg} \cdot \text{m}^2)(2.00 \times 10^{12}\ \text{rad/s})^2/2 = 3.89 \times 10^{-22}\ \text{J}$.