Columbia University
Department of Physics
QUALIFYING EXAMINATION

Wednesday, January 14, 2015
1:00PM to 3:00PM
Modern Physics
Section 3. Quantum Mechanics

Two hours are permitted for the completion of this section of the examination. Choose 4 problems out of the 5 included in this section. (You will not earn extra credit by doing an additional problem). Apportion your time carefully.

Use separate answer booklet(s) for each question. Clearly mark on the answer booklet(s) which question you are answering (e.g., Section 3 (QM), Question 2, etc.).

Do **NOT** write your name on your answer booklets. Instead, clearly indicate your **Exam Letter Code**.

You may refer to the single handwritten note sheet on $8\frac{1}{2}$" $\times$ 11" paper (double-sided) you have prepared on Modern Physics. The note sheet cannot leave the exam room once the exam has begun. This note sheet must be handed in at the end of today's exam. Please include your Exam Letter Code on your note sheet. No other extraneous papers or books are permitted.

Simple calculators are permitted. However, the use of calculators for storing and/or recovering formulae or constants is NOT permitted.

Questions should be directed to the proctor.

Good Luck!

1. In a region $x > 0$, there is a uniform magnetic field $B_z = B$ and in the region $x < 0$, $B_z = 0$. A beam of polarized neutrons of mass m, spin $s = 1/2$, magnetic moment μ_0 and total energy E approach the plane $x = 0$ from the field free region along the x-axis, and thus hit the plane at normal incidence. Show that the reflection coefficient at the $x = 0$ plane depends on the incident polarization of the neutrons, assumed to be either parallel or anti-parallel to the B-field. Find an explicit expression for the reflection coefficient from the $x = 0$ plane for neutrons with these two polarizations. Consider both the case $E > \mu_0 B$ and $E < \mu_0 B$.

2. The Hamiltonian of a harmonic oscillator can be written in dimensionless units as

$$\hat{H} = \frac{1}{2}(\hat{p}^2 + \hat{x}^2) \equiv \frac{1}{2} + a^\dagger a$$

where the operators $a^\dagger$ and a are

$$\hat{a} = \frac{1}{\sqrt{2}}(\hat{x} + i\hat{p}) \quad \hat{a}^\dagger = \frac{1}{\sqrt{2}}(\hat{x} - i\hat{p})$$

The normalized ground state wave function of $\hat{H}$ may be written as

$$\psi(x) = \langle x|0\rangle = \frac{1}{\pi^{1/4}}e^{-\frac{x^2}{2}} \text{ or as } \psi(p) = \langle p|0\rangle = \frac{1}{\pi^{1/4}}e^{-\frac{p^2}{2}}$$

and the n$^{\text{th}}$ excited state is $(a^\dagger)^n|0\rangle/\sqrt{n!}$.

Suppose that at negative times the oscillator is in its ground state and that at time $t = 0$ an electric field is suddenly turned on so the oscillation is subject to an additional potential $V(x) = Cx$ so the Hamiltonian becomes

$$\hat{H}_{\text{new}} = \frac{1}{2}(\hat{p}^2 + \hat{x}^2) + C\hat{x}$$

(a) Find the expectation value of the energy immediately after the electric field is turned on

(b) Let $\hat{U}(C) = e^{i\hat{p}C}$. Show that the Hamiltonian after the field is turned on is related to the original Hamiltonian as

$$\hat{H} + C\hat{x} = \hat{U}(C)\hat{H}\hat{U}^\dagger(C) - \frac{C^2}{2}$$

(c) Show that immediately after the perturbation is turned on, the probability that the system is in the ground state of $\hat{H}_{\text{new}}$ is

$$P_0 = e^{-\frac{C^2}{2}}$$

(d) Find the probability that the particle will be in the n$^{\text{th}}$ eigenstate of $\hat{H}_{\text{new}}$ at time $t > 0$.

3. Consider a spin-1/2 particle with spin operator $\vec{s}$ and magnetic moment $\mu = \gamma\vec{s}$ placed in a time-independent magnetic field pointing in the z-direction: $\vec{B} = B\hat{z}$.

   (a) If at $t = 0$ the particle's spin in the x-direction is measured and found to be $+\hbar/2$, what is the quantum state of the system, expressed in the basis of eigenstates of s_z?

   (b) If the state resulting from the measurement in part (a) evolves with time, what is the resulting state $|\psi(t)\rangle$ at the time t?

   (c) Find the average values for the particles spin in the y-direction at the time t.

   (d) If the spin of the state $|\psi(t)\rangle$ is measured in the $\hat{z}\cos(\theta) + \hat{x}\sin(\theta)$ direction and the result $-\hbar/2$ obtained, what will be the resulting quantum state?

   (e) What is the probability that the value $-\hbar/2$ will be found?

4. Consider a two-dimensional harmonic oscillator described by the Hamiltonian

$$H = \frac{1}{2m}(p_x^2 + p_y^2) + \frac{m\omega^2}{2}(x^2 + y^2)$$

Let $J = \langle 1|x|0\rangle$, where $|0\rangle$ and $|1\rangle$ are states of the one-dimensional harmonic oscillator with frequency ω

(a) What are the unperturbed eigenvalues and eigenstates of energy?

(b) Calculate the shift of the ground state energy, up to and including terms of order λ^2, due to the perturbation $V = \lambda xy$. You may express your answer in terms of J.

(c) Using degenerate first-order perturbation theory, find the leading shifts in the energies of the degenerate first excited states. Again, you may express your answer in terms of J.

5. (a) The process of "K-capture" involves the capture of an inner orbital electron by the nucleus, resulting in a reduction of the nuclear charge by one unit. This process is in part due to the non-zero probability that an electron can be found within the volume of the nucleus. An electron is in the 1s state of a hydrogenic potential, with a wave function given by

$$\psi(r) = \frac{1}{\sqrt{\pi}} \left(\frac{Z}{a_0} \right)^{3/2} e^{-Zr/a_0}, \tag{1}$$

where Z is the atomic number and a_0 is the Bohr radius.

Calculate the probability that a 1s electron will be found within the nucleus. Take the nuclear radius to be $R = 10^{-5}\mathring{A}$, which you may approximate as being much smaller than the Bohr radius.

(b) Assume than an electron is initially in a hydrogenic ground state described in equation (1) with $Z = 2$. A nuclear reaction abruptly changes the nuclear charge to $Z = 1$. What is the probability that the electron will be found in the ground state of the new potential after the change in nuclear charge?

(c) Assume that instead of the final state described in part (b), the nuclear reaction leaves the electron in a state given by

$$\Psi(r, \theta, \phi) = A(\sin\theta \sin\phi + \sin\theta\cos\phi + \cos\theta) r e^{-r/a_0},$$

where A is a constant such that $|\Psi|^2$ is normalized to unity.

What are the possible values that can be obtained in measurements of L^2 and L_z, and with what probabilities will these values be measured?

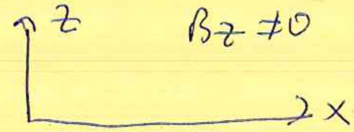
Recall that the first several spherical harmonic $Y_{z,m}$ can be written

$$Y_{0,0} = \frac{1}{\sqrt{4\pi}};\ Y_{1,1} = -\sqrt{\frac{3}{8\pi}} e^{i\phi} \sin\theta;\ Y_{1,0} = \sqrt{\frac{3}{4\pi}} \cos\theta$$

$$Y_{2,2} = \sqrt{\frac{15}{32\pi}} e^{2i\phi} \sin^2\theta;\ Y_{2,1} = \sqrt{\frac{5}{16\pi}} (3\cos^2\theta - 1)$$

Hailey
QM soln.

$B_z = 0$ $\quad\quad$ $B_z \neq 0$



E, neutrons $\dashrightarrow$

$$V = -\vec{\mu}\cdot\vec{B} = -\mu_0 B_z \sigma_z$$

Absorbing $g, \hbar$ into μ_0

$$\psi_{\uparrow\downarrow} = u(x)\chi_{\uparrow\downarrow}\ ; \quad \sigma_z \chi_\uparrow = \chi_\uparrow \quad \sigma_z\chi_\downarrow = -\chi_\downarrow$$

$x<0$ $\quad u_{\uparrow\downarrow} = a e^{ik_0 x} + b_{\uparrow\downarrow} e^{-ik_0 x}$

$x>0$ $\quad u_{\uparrow\downarrow} = c e^{ik_{\uparrow\downarrow} x}$

For spin up $x>0$

$$-\frac{\hbar^2}{2m} u'' \chi_\uparrow - \mu_0 B\sigma_z \chi_\uparrow u = Eu$$

$$u'' + k_\uparrow^2 u = 0 \quad k_\uparrow^2 = \frac{2m(E+\mu_0 B)}{\hbar^2}$$

$x<0$ $\quad -\hbar^2/_{2m}\, u'' = Eu \Rightarrow k_0^2 = \frac{2m}{\hbar^2}E$

Matching at $x=0$ $\quad a + b_\uparrow = c_\uparrow$

$$ik_0(a-b_\uparrow) = ik_\uparrow c_\uparrow$$

$$\Rightarrow \quad b_\uparrow/a = \frac{k_0-k_\uparrow}{k_0+k_\uparrow}$$

For spin down we will clearly get same thing

$$b_\downarrow/a = \frac{k_0-k_\downarrow}{k_0+k_\downarrow} \quad \text{but with}$$

$k_\downarrow^2 = \frac{2m}{\hbar^2}(E - \mu_0 B)$ from $\sigma_z\chi_\downarrow$ term in V

So $\quad R_{\uparrow\downarrow} = \left(\frac{k_0-k_{\uparrow\downarrow}}{k_0+k_{\uparrow\downarrow}}\right)^2$

The above is for $E>\mu_0 B$. For $E<\mu_0 B$ the $\downarrow$ polarization will be 100% reflected and the $\uparrow$ polarization will yield the same answer as for $E>\mu_0 B$

Solution: Harmonic Oscillator in Electric Field

1. Let $|0>$ be the ground state of $\hat{H}$. Then

$$E =< 0|\hat{H}_{new}|0 >=< 0|\hat{H} + C\hat{x}|0 > \frac{1}{2} + < 0\hat{x}|0 >= \frac{1}{2}$$

2. $e^{-iX\hat{p}}$ is the translation operator that shifts x by X and commutes with $\hat{p}$, so

$$\begin{aligned}\hat{U}(C)\hat{H}_{new}\hat{U}^{\dagger}(C) &= \frac{\hat{p}^2}{2} + \frac{(\hat{x}-C)^2}{2} + C(\hat{x}-C) \\ &= \frac{\hat{p}^2}{2} + \frac{\hat{x}^2}{2} - \frac{C^2}{2}\end{aligned}$$

3. The probability is

$$\begin{aligned}|< 0_{new}|0 >|^2 &= \left|\int dx \frac{e^{-\frac{(x+C)^2}{2}}}{\pi^{\frac{1}{4}}}\frac{e^{-\frac{x^2}{2}}}{\pi^{\frac{1}{4}}}\right|^2 \\ &= \left|\int \frac{dx}{\sqrt{\pi}} e^{-x^2 - Cx - \frac{C^2}{2}}\right|^2 \\ &= \left|\int \frac{dx}{\sqrt{\pi}} e^{-(x+\frac{C}{2})^2 - \frac{C^2}{4}}\right|^2 \\ &= e^{-\frac{C^2}{2}}\end{aligned}$$

4. *This may be asking too much. Could give the disentangling formula (representation of U below) as a hint or in the specification of the problemt*

   The required probability is

$$Prob_n = |< 0_{new}|\frac{a^n}{\sqrt{n!}}|0 >|^2 = |< 0_{new}|\frac{a^n}{\sqrt{n!}}U^{\dagger}(C)|0_{new} >|^2$$

   Using the representation $U^{\dagger}(C) = e^{-\frac{C^2}{4}} e^{-\frac{i}{\sqrt{2}}a^{\dagger}} e^{\frac{i}{\sqrt{2}}a}$, noting that $e^{\frac{i}{\sqrt{2}}a} = 0$ and taking the n^{th} order term in the expansion of $e^{-\frac{i}{\sqrt{2}}a^{\dagger}}$ and using the normalization of the eigenstate we find that

$$Prob_n = e^{-\frac{C^2}{2}}|< 0_{new}|\frac{a^n}{\sqrt{n!}}\frac{1}{n!}\left(\frac{-iCa^{\dagger}}{\sqrt{2}}\right)^n|0_{new} >|^2 = \frac{\left(\frac{C^2}{2}\right)^n}{n!}e^{-\frac{C^2}{2}}$$

1. Consider a spin-1/2 particle with spin operator $\vec{s}$ and magnetic moment $\mu = \gamma\vec{s}$ placed in a time-independent magnetic field pointing in the z-direction: $\vec{B} = B\hat{z}$.
   (a) If at $t = 0$ the particle's spin in the x-direction is measured and found to be $+\hbar/2$, what is the quantum state of the system, expressed in the basis of eigenstates of s_z?
   (b) If the state resulting from the measurement in part (a) evolves with time, what is the resulting state $|\psi(t)\rangle$ at the time t?
   (c) Find the average values for the particle's spin in the y-direction at the time t.
   (d) If the spin of the state $|\psi(t)\rangle$ is measured in the $\hat{z}\cos(\theta) + \hat{x}\sin(\theta)$ direction and the result $-\hbar/2$ obtained, what will be the resulting quantum state?
   (e) What is the probability that the value $-\hbar/2$ will be found?

Solution:

(a) The eigenstates of σ_x are $(|+\rangle \pm |-\rangle)/\sqrt{2}$ with eigenvalues $\pm\hbar/2$. The basis states $|\pm\rangle$ are eigenstates of s_z with eigenvalues $\pm\hbar/2$. Therefore, if s_x is found to be $+\hbar/2$ the quantum state must be $(|+\rangle + |-\rangle)/\sqrt{2}$.

(b) The Hamiltonian for the system is $H = -\vec{\mu}\cdot\vec{B} = -\gamma s_z B$ so:

$$|\psi(t)\rangle = e^{-iHt/\hbar}|\psi(0)\rangle = \frac{1}{\sqrt{2}}\left(e^{+i\gamma Bt/2}|+\rangle + e^{-i\gamma Bt/2}|-\rangle\right)$$

(c) Evaluate the expectation value:

$$\langle s_y\rangle = \langle\psi(t)|s_x|\psi(t)\rangle = \frac{\hbar}{4}\begin{pmatrix} e^{-i\gamma Bt/2} & e^{+i\gamma Bt/2}\end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0\end{pmatrix}\begin{pmatrix} e^{+i\gamma Bt/2} \\ e^{-i\gamma Bt/2}\end{pmatrix} = -\frac{\hbar}{2}\sin(\gamma Bt)$$

(d) The eigenstates of $s_z\cos(\theta) + s_x\sin(\theta)$ can be found by direct diagonalization or by rotating $|\pm\rangle$ through θ about the y-axis:

$$e^{-is_y\theta/\hbar}|\pm\rangle = \big(\cos(\theta/2) - i\sigma_y\sin(\theta/2)\big)|\pm\rangle = \cos(\theta/2)|\pm\rangle \pm \sin(\theta/2)|\mp\rangle.$$

so if the result $-\hbar/2$ is found the resulting state will be the eigenfunction $\cos(\theta/2)|-\rangle - \sin(\theta/2)|+\rangle$.

(e) The probability of finding $-\hbar/2$ is then

$$\begin{aligned}
&\left|\frac{1}{\sqrt{2}}\left(e^{-i\gamma Bt/2}\langle +| + e^{+i\gamma Bt/2}\langle -|\right)\left(\cos(\theta/2)|-\rangle - \sin(\theta/2)|+\rangle\right)\right|^2 \\
&\qquad = \frac{1}{2}\Big(\cos^2(\gamma Bt/2)\big(\cos(\theta/2) - \sin(\theta/2)\big)^2 + \sin^2(\gamma Bt/2)\big(\cos(\theta/2) + \sin(\theta/2)\big)^2\Big) \quad (1) \\
&\qquad = \frac{1}{2}\big(1 + \cos(\gamma Bt)\sin(\theta)\big).
\end{aligned}$$

Weinberg,
QM problem

a) The unperturbed states are $|n_x, n_y\rangle$, with
$E = (n_x + n_y + 1)\hbar\omega$ and $n_x, n_y = 0, 1, 2 \ldots$

b) The ground state is $|0,0\rangle$. The order-λ shift is

$$E^{(1)} = \lambda \langle 00 | x y | 00 \rangle$$
$$= 0 \quad \text{(odd integrals)}$$

The order λ^2 shift is

$$E^{(2)} = \sum_{n_x, n_y} \frac{|\langle n_x, n_y | V | 00 \rangle|^2}{E_{00} - E_{n_x, n_y}}$$

The only nonzero matrix element is
$$\lambda \langle 11 | xy | 00 \rangle = \lambda J^2$$

$$\Rightarrow E^{(2)} = -\frac{\lambda^2 J^4}{2\hbar\omega}$$

c) The degenerate states are $|10\rangle$ and $|01\rangle$

$$\langle 01 | V | 01 \rangle = \langle 10 | V | 10 \rangle = 0$$

$$\langle 01 | V | 10 \rangle = \langle 10 | V | 01 \rangle = \lambda J^2$$

The energy shifts are the eigenvalues of

$$\begin{pmatrix} 0 & \lambda J^2 \\ \lambda J^2 & 0 \end{pmatrix}$$

which are $\pm \lambda J^2$

Solution:

(a) Because $R \ll a_0 \approx 0.53\AA$, the wave function can be approximated by its value at the origin for $r < R$, such that

$$P(r < R) = \int_{r<R} \Psi^* \Psi dV \approx (4\pi/3)R^3|\Psi(r=0)|^2 = \frac{4}{3}Z^3 \left(\frac{R}{a_0}\right)^3 \approx 10^{-14}Z^3 \tag{4}$$

(b) Assuming the nuclear transition is abrupt, the initial wave function of the electron immediately after the transition $\Psi(Z=2)$ will be identical to the ground state (eq. 1) with $Z=2$. The probability to be found in the ground-state of the new nucleus $\Psi(Z=1)$ is thus given by

$$\langle \Psi_{Z=2}|\Psi_{Z=1}\rangle = 4\pi \int_0^\infty \Psi^*_{Z=2}(r)\Psi_{Z=1}(r)r^2 dr \approx \frac{2^{7/2}}{a_0^3}\int_0^\infty r^2 e^{-3r/a_0} dr \approx \frac{2^{9/2}}{27} \approx 0.838 \tag{5}$$

$$|\langle \Psi_{Z=2}|\Psi_{Z=1}\rangle|^2 = 0.838^2 = 0.702 \tag{6}$$

(c) The angular part of the wave function can be decomposed into spherical harmonics as

$$\Psi \propto \sqrt{\frac{3}{4\pi}}\left\{\frac{1}{\sqrt{2}}\left[\Upsilon_{1,1}(1-i) - \Upsilon_{1,-1}(1+i)\right] + \Upsilon_{1,0}\right\}, \tag{7}$$

Now using the fact that the $\Upsilon_{l,m}$'s are eigenfunctions of the angular momentum operators,

$$\langle \Upsilon_{l,m}|L^2|\Upsilon_{l,m}\rangle = l(l+1)\hbar^2; \quad \langle \Upsilon_{l,m}|L_z|\Upsilon_{l,m}\rangle = m\hbar, \tag{8}$$

the only possible value of L^2 is $2\hbar^2$, while $L_z = 0, 1, -1$ each occurs with probability of $1/3$, based on the moduli of the coefficients in equation 7.