# Columbia University
# Department of Physics
# QUALIFYING EXAMINATION

Monday, January 12, 2015
3:10PM to 5:10PM
Classical Physics
Section 2. Electricity, Magnetism & Electrodynamics

Two hours are permitted for the completion of this section of the examination. Choose 4 problems out of the 5 included in this section. (You will not earn extra credit by doing an additional problem). Apportion your time carefully.

Use separate answer booklet(s) for each question. Clearly mark on the answer booklet(s) which question you are answering (e.g., Section 2 (Electricity etc.), Question 2, etc.).

Do **NOT** write your name on your answer booklets. Instead, clearly indicate your **Exam Letter Code**.

You may refer to the single handwritten note sheet on $8\frac{1}{2}$" $\times$ 11" paper (double-sided) you have prepared on Classical Physics. The note sheet cannot leave the exam room once the exam has begun. This note sheet must be handed in at the end of today's exam. Please include your Exam Letter Code on your note sheet. No other extraneous papers or books are permitted.

Simple calculators are permitted. However, the use of calculators for storing and/or recovering formulae or constants is NOT permitted.

Questions should be directed to the proctor.

Good Luck!

1. A region in space contains a total positive charge $Q$ that is distributed spherically such that the volume charge density $\rho(r)$ is given by:

$$\begin{aligned}\rho(r) &= \alpha \text{ for } r \leq R/2 \\ \rho(r) &= 2\alpha(1 - r/R) \text{ for } R/2 \leq r \leq R \\ \rho(r) &= 0 \text{ for } r \geq R\end{aligned}$$

where $\alpha$ is a positive constant having units of $\mathrm{C/m^3}$.

(a) Determine $\alpha$ in terms of $Q$ and $R$.

(b) Derive an expression for the magnitude of the electric field $\vec{E}(r)$ for $r \leq R/2$.

(c) Derive an expression for the magnitude of the electric field $\vec{E}(r)$ for $R/2 \leq r \leq R$.

(d) Derive an expression for the magnitude of the electric field $\vec{E}(r)$ for $r \geq R$.

(e) What fraction of the total charge is contained within the region $r \leq R/2$.

(f) If an electron with charge $q' = -e$ and mass $m$ is oscillating back and forth radially through $r = 0$ (the center of the distribution) with an amplitude less than $R/2$, describe qualitatively the nature of the oscillation.

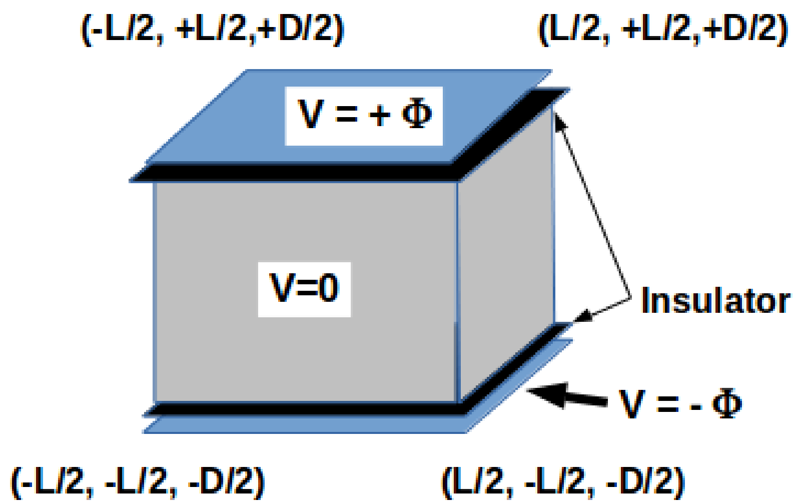
(g) What is the period of the motion in part f)?

(h) If the amplitude of the motion described in part f) is greater than $R/2$, describe qualitatively the nature of the oscillation.

2.

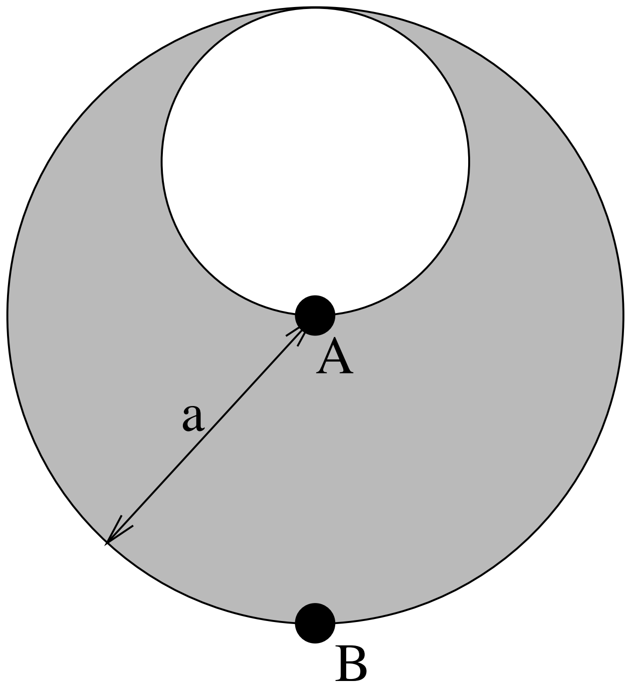
(a) Calculate the electrostatic potential $\phi(x, y, z)$ in a conducting box with boundary condition $\phi(x, y, \pm D/2) = \pm\Phi$ on top and bottom and grounded $\phi(\pm L/2, \pm L/2, z) = 0$ on the four sides insulated from the top and bottom plates.

(b) Check that you recover the same idealized capacitor result in the $L \to \infty$ limit.



3. A perfectly conducting sphere of radius $R$ moves with constant velocity $\vec{v} = v\hat{x}$ through a uniform magnetic field $\vec{B} = B_0\hat{y}$. Assuming that $v \ll c$, find the surface charge density induced on the sphere to lowest nontrivial order in $v/c$.

4. A long cylindrical conductor of radius $a$ has an off-center cylindrical hole of radius $a/2$ down its full length, as shown in the figure. If a current $I$ flows through the conductor into the page, then what is the strength and direction of the magnetic field at $A$? What is it at $B$?



5. A static magnetic monopole (i.e., a point magnetic charge giving rise to a Coulomb magnetic field) is located at the origin and a static electric charge is located on the $z$-axis at $(0, 0, a)$.

   (a) Write expression for the magnetic and electric fields in terms of the magnetic charge $g$ and electric charge $q$.

   (b) Even though this is a static situation, these magnetic and electric fields have net angular momentum. Show that the magnitude of this angular momentum is independent of the distance $a$ between the charges, and determine the direction in which this angular momentum points. Note that you do not have to determine the magnitude of the angular momentum, but only show that it is independent of $a$.

   (c) Suppose that the electric charge is allowed to move non-relativistically. Show that the usual expression for its angular momentum, $\mathbf{r} \times (m\mathbf{v})$, is not conserved. Find a vector $\mathbf{G}$ such that $\mathbf{r} \times (m\mathbf{v}) + \mathbf{G}$ is conserved.

2015 Quals Question: E&M (Dodd)

Problem:

A region in space contains a total positive charge *Q* that is distributed spherically such that the volume charge density *ρ*(*r*) is given by:

$\rho(r) = \alpha$ for $r \le R/2$

$\rho(r) = 2\alpha(1-r/R)$ for $R/2 \le r \le R$

$\rho(r) = 0$ for $r \ge R$

where α is a positive constant having units of C/m$^3$.

a). Determine α in terms of *Q* and *R*.

b). Derive an expression for the magnitude of the electric field $\vec{E}(r)$ for $r \le R/2$.

c). Derive an expression for the magnitude of $\vec{E}(r)$ for $R/2 \le r \le R$.

d). Derive an expression for the magnitude of $\vec{E}(r)$ for $r \ge R$.

e). What fraction of the total charge is contained within the region $r \le R/2$?

f). If an electron with charge q'=−e and mass m is oscillating back and forth about r=0 (the center of the distribution) with an amplitude less than R/2, describe qualitatively the nature of the oscillation.

g). What is the period of the motion in part f)?

h). If the amplitude of the motion described in part f) is greater than *R*/2, describe qualitatively the nature of the oscillation.

Solution:

Use a Gaussian surface that is a sphere of radius *r*. Let $Q_i$ be the charge in the region $r \le R/2$ and let $Q_0$ be the charge in the region where $R/2 \le r \le R$.

a). The charge in a spherical shell of radius *r* and thickness *dr* is $dQ = \rho(r)4\pi r^2\, dr$. Apply Gauss's law.

The total charge is $Q = Q_i + Q_0$, where $Q_i = \alpha\dfrac{4\pi(R/2)^3}{3} = \dfrac{\alpha\pi R^3}{6}$ and

$$Q_0 = 4\pi(2\alpha)\int_{R/2}^{R}(r^2 - r^3/R)\, dr = 8\alpha\pi\left(\frac{(R^3 - R^3/8)}{3} - \frac{(R^4 - R^4/16)}{4R}\right) = \frac{11\alpha\pi R^3}{24}.$$

Therefore, $Q = \dfrac{15\alpha\pi R^3}{24}$ and $\alpha = \dfrac{8Q}{5\pi R^3}$.

b). For $r \le R/2,$ Gauss's law gives $E 4\pi r^2 = \dfrac{\alpha 4\pi r^3}{3e_0}$ and $E = \dfrac{\alpha r}{3e_0} = \dfrac{8Qr}{15\pi e_0 R^3}.$

c). For $R/2 \le r \le R,$ $E 4\pi r^2 = \dfrac{Q_i}{e_0} + \dfrac{1}{e_0}\left(8\alpha\pi\left(\dfrac{(r^3 - R^3/8)}{3} - \dfrac{(r^4 - R^4/16)}{4R}\right)\right)$ and

$$E = \frac{\alpha\pi R^3}{24e_0(4\pi r^2)}(64(r/R)^3 - 48(r/R)^4 - 1) = \frac{kQ}{15r^2}(64(r/R)^3 - 48(r/R)^4 - 1).$$

d). For $r \ge R,$ $E(4\pi r^2) = \dfrac{Q}{e_0}$ and $E = \dfrac{Q}{4\pi e_0 r^2}.$

e). $\dfrac{Q_i}{Q} = \dfrac{(4Q/15)}{Q} = \dfrac{4}{15} = 0.267.$

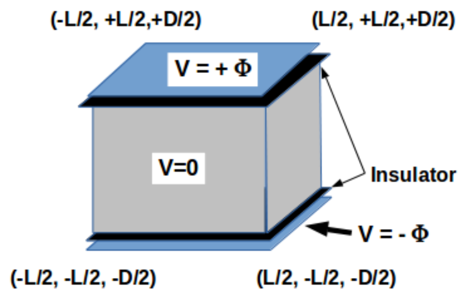
f). For $r \le R/2,$ $F_r = -eE = -\dfrac{8eQ}{15\pi e_0 R^3}r,$ so the restoring force depends upon displacement to the first power, and we have simple harmonic motion.

g). Comparing to $F = -kr, k = \dfrac{8eQ}{15\pi e_0 R^3}.$ Then $\omega = \sqrt{\dfrac{k}{m_e}} = \sqrt{\dfrac{8eQ}{15\pi e_0 R^3 m_e}}$ and $T = \dfrac{2\pi}{\omega} = 2\pi\sqrt{\dfrac{15\pi e_0 R^3 m_e}{8eQ}}.$

h). If the amplitude of oscillation is greater than $R/2,$ the force is no longer linear in $r,$ and is thus no longer simple harmonic.

# Quals 2009: Electromagnetism (M.Gyulassy)

1. a) Calculate the electrostatic potential $\phi(x, y, z)$ in a conductor box with boundary condition $\phi(x, y, \pm D/2) = \pm\Phi$ on top and bottom and grounded $\phi(\pm L/2, \pm L/2, z) = 0$ on the four sides insulated from the top and bottom plates. (b) Check that you recover the sime idealized capacitor result in the $L \to \infty$ limit.




Solution

Use separarion of variable $\phi = X(x)Y(y)Z(z)$ with $d^2/dx^2$ normalized eigenstate $X(x) = \sqrt{2/L}\cos((2m+1)\pi x/L)$ and $d^2/dy^2$ eigenstate $Y(x) = \sqrt{2/L}\cos((2n+1)\pi y/L)$ with $n, m = 0, \cdots, \infty$ required to satisfy the grounded side BC. This leaves us to solve for each $n, m$

$$\frac{d^2}{dz^2}Z_{n,m+\lambda_{n,m}Z_{n,m}=0}$$

with BC $Z_{n,m}(\pm D/2) = \pm\Phi$. Since $\lambda_{n,m} = (2\pi/L)((2n+1)^2 + (2m+1)^2) \equiv k^2_{n,m}$, this is soved by $Z = \sinh(\sqrt{\lambda_{n,m}}z)$ and thus the solution can be expanded as
$\phi(x, y, z) = \sum_{n,m=0}^{\infty} c_{n,m}X_n(x)Y_m(y)\sinh(k_{n,m}z)$
We can extract $c_{n,m}$ by integrating $\int dxdyX_mY_n\phi(x, y, D/2) = c_{n,m}\sinh(k_{n,m}D/2)$. The left hand side is $\Phi(-1)^{n+m}(2/L)(4L/\pi)^2/((2n+1)(2m+1))$ and therefore

$$\phi(x, y, z) = \Phi\frac{8L}{\pi^2}\sum_{n,m=0}^{\infty}(-1)^{n+m}\frac{X_n(x)}{(2n+1)}\frac{Y_m(y)}{(2m+1)}\frac{\sinh(k_{n,m}z)}{\sinh(k_{n,m}D/2)}$$

(b) For $L \to \infty$ , $k_{n,m}$ becomes very small for any fixed $m, n$ allowing the expansion $\sinh(x) \approx x$. Thw ratio of sinh terms reduces to $2z/D$ and can be pulled out of the sum. The independent sums over n and m correspond to the fourier representation of $\theta(L/2 - |x|)\theta(L/2 - |y|)$. Therefore within the capacitor we recover the idealized $\phi = \Phi\ (z/(D/2))$ linear in $z$ formula independent of $x$ and $y$ and a constant $E_z = -2\Phi/D$ electric field.

**Solution: Sphere moving in uniform magnetic field**

Lorentz transform to the rest frame of the sphere. The change in shape of the sphere is of order $(v/c)^2$ and may be neglected.

In this frame the surface of the sphere must be an equipotential, and the field inside the sphere must vanish.

Outside of the sphere we have (denoting the fields in the rest frame of the sphere with primes and working to lowest order in $v/c$)

$$\vec{E}' = -\frac{vB}{c}\hat{z} + \mathcal{O}\left(\frac{v}{c}\right)^2 \quad (1)$$

Also to leading order in $\frac{v}{c}$ the shape of the sphere is not changed.

The charge density is then fixed by the condition (from $\vec{\nabla} \cdot \vec{E} = 4\pi\rho$) that the surface charge density is $1/(4\pi)$ times the change across the interface of the normal component of the electric field, so

$$\rho' = -\frac{v_B cos\theta}{4\pi c} \quad (2)$$

and to order $v/c$ there is no diamagnetism so the current is zero.

Because the charge is already of order $v/c$ the order $v/c$ contribution in the moving frame is the same.

# 1 E&M Problem

A long cylindrical conductor of radius $a$ has an off-center cylindrical hole of radius $a/2$ down its full length, as shown in Figure 1. If a current $I$ flows through the conductor into the page, then what is the strength and direction of the magnetic field in $A$? What is it in $B$?

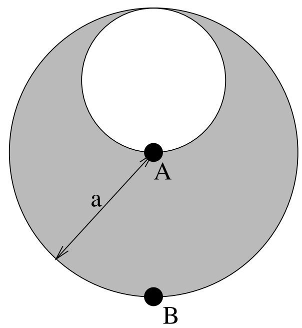



Figure 1: Cross section of a long cylindrical conductor with an off-center cylindrical hole.

## 1.1 Solution

Use the principle of superposition: this is the sum of a conductor of radius $a$ with the current flowing into the page, and a conductor of radius $a/2$ with current flowing out of the page.

The current density is

$$J = \frac{I}{\pi a^2 - \pi (a/2)^2} = \frac{4I}{3\pi a^2}. \tag{1}$$

Let's start with point $A$: for the contribution from the large cylinder draw an Amperian loop at an arbitrary radius inside the cylinder. Then $\oint B \cdot dl = \mu_0 I_{enc}$ yields $B 2\pi s = \mu_0 J \pi s^2$ and

$$B(s) = \frac{2\mu_0 I}{3\pi a^2} s\hat{\phi}. \tag{2}$$

At $s = 0$, $B = 0$. For the small cylinder, a loop around the circumference of the cylinder gives $B2\pi(a/2) = \mu_0 J\pi(a/2)^2$ so that

$$B = \frac{\mu_0 I}{3\pi a}(-\hat{\phi}), \tag{3}$$

which is thus the total field at $A$.

For the contribution from the large cylinder to point B, take a loop around the circumference, That gives

$$B = \frac{2\mu_0 I}{3\pi a}\hat{\phi}. \tag{4}$$

The contribution from the small cylinder requires drawing an Amperian loop centered on the small cylinder's axis with radius $3a/2$. That yields

$$B = \frac{\mu_0 I}{9\pi a}(-\hat{\phi}), \tag{5}$$

so that the total field at point $B$ is

$$B_{tot} = \frac{5\mu_0 I}{9\pi a}\hat{\phi}. \tag{6}$$

Weinberg,
E&M problem

a) $\vec{B} = \frac{g}{4\pi} \frac{\vec{r}}{r^3}$

$\vec{E} = \frac{q}{4\pi} \frac{(\vec{r} - a\hat{z})}{|\vec{r} - a\hat{z}|^3}$

b) $L_{field} = \int d^3r \, \vec{r} \times (\vec{E} \times \vec{B})$

From symmetry, $L_x$ and $L_y$ must vanish

$(L_{field})_z = \int d^3r \, \vec{r} \times (\vec{E} \times \vec{B}) \cdot \hat{z}$

$\vec{r} \times (\vec{E} \times \vec{B}) \cdot \hat{z} = \vec{E} \cdot \hat{z} \, \vec{r} \cdot \vec{B} - \vec{B} \cdot \hat{z} \, \vec{r} \cdot E$

$= \frac{gq}{(4\pi)^2} \frac{1}{r^3} \frac{1}{|r - a\hat{z}|^3} \{ (z-a) r^2 - z(r^2 - az) \}$

$= \frac{gq}{(4\pi)^2} \frac{1}{r^3} \frac{1}{|\vec{r} - a\hat{z}|^3} \{ a(z^2 - r^2) \}$

$= \frac{gq}{(4\pi)^2} \frac{(-a \sin^2\theta)}{r} \frac{1}{|\vec{r} - a\hat{z}|^3}$

Now let $\vec{r} = a\vec{s}$

Then

$\vec{r} \times (\vec{E} \times \vec{B}) \cdot \hat{z} = -\frac{1}{a^3}$ (positive function of $\vec{s}$)

where the (pos. fn.) is indep of $a$

$d^3r = a^3 d^3s$

$\Rightarrow \vec{L}_{field}$ is independent of $a$, and in the $-\hat{z}$ direction; i.e., it points from the electric to the magnetic charge

c) The electric charge feels a force

$$\vec{F} = q\,\vec{v}\times\vec{B} = q\left(\frac{g}{4\pi}\right)\vec{v}\times\frac{\vec{r}}{r^3}$$

If $\vec{L}_{orb} = \vec{r}\times m\vec{v}$,

$$\frac{d\vec{L}_{orb}}{dt} = \frac{d\vec{r}}{dt}\times m\vec{v} + \vec{r}\times\frac{d(m\vec{v})}{dt} = \vec{r}\times\vec{F}$$

$$= \frac{qg}{4\pi}\,\vec{r}\times(\vec{v}\times\vec{r})\left(\frac{1}{r^3}\right)$$

$$= \frac{qg}{4\pi}\frac{1}{r^3}\left[\vec{v}\,r^2 - (\vec{r}\cdot\vec{v})\,\vec{r}\right]$$

$$= \frac{qg}{4\pi}\left[\frac{\vec{v}}{r} - \frac{\vec{r}\cdot\vec{v}}{r^2}\right]$$

$$= \frac{qg}{4\pi}\frac{d}{dt}(\hat{r})$$

$\Rightarrow\quad \vec{r}\times m\vec{v} - \frac{qg}{4\pi}\hat{r}$ is conserved