Columbia University
Department of Physics
QUALIFYING EXAMINATION

Wednesday, January 13, 2016
3:10PM to 5:10PM
Modern Physics
Section 4. Relativity and Applied Quantum Mechanics

Two hours are permitted for the completion of this section of the examination. Choose 4 problems out of the 5 included in this section. (You will not earn extra credit by doing an additional problem). Apportion your time carefully.

Use separate answer booklet(s) for each question. Clearly mark on the answer booklet(s) which question you are answering (e.g., Section 4 (Relativity and Applied QM), Question 2, etc.).

Do **NOT** write your name on your answer booklets. Instead, clearly indicate your **Exam Letter Code**.

You may refer to the single handwritten note sheet on $8\frac{1}{2}$" $\times$ 11" paper (double-sided) you have prepared on Modern Physics. The note sheet cannot leave the exam room once the exam has begun. This note sheet must be handed in at the end of today's exam. Please include your Exam Letter Code on your note sheet. No other extraneous papers or books are permitted.

Simple calculators are permitted. However, the use of calculators for storing and/or recovering formulae or constants is NOT permitted.

Questions should be directed to the proctor.

Good Luck!

1. Consider the first relativistic correction, H_1, in the Hamiltonian for the one dimensional harmonic oscillator.

$$H = H_0 + H_1 = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2 - \frac{1}{2mc^2}\left(\frac{p^2}{2m}\right)^2 \tag{1}$$

Evalulate the first order shift in the ground state energy due to H_1.
Recall that the ground state wave function is given by

$$U_0(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \exp\left[-\frac{m\omega x^2}{2\hbar}\right] \tag{2}$$

2. An asteroid is on a collision course with a space station located 5000 light-minutes from Earth. The asteroid is moving away from the Earth toward the space station at a speed of 3/5c along a trajectory which is a straight line connecting Earth and the space station. To save the station, NASA launches a missle from Earth at 4/5c. When the missile is launched, NASA determines that the asteroid is 400 light-minutes from Earth.

   (a) How many minutes should NASA set on a timer located on the missle so that it will explode just as it catches up to the asteroid? (In all parts of this problem, ignore subtleties having to do with acceleration.)

   (b) A few weeks later, another asteroid is on a similar collision course with the space station, travelling again at 3/5c. NASA decides to send another missile to destroy it, but this time they want the missile to pass the asteroid and explode only when according to sensors on the missile, the missile is 350 light minutes beyond the asteroid. In this scenario, how many minutes should NASA set on the missile's timer?

   (c) From the perspective of the missile's frame of reference, how far away is the asteroid when the missile is launched?

3. A thin plate with surface rest-mass density Σ_0 $[g/cm^2]$ is surrounded by uniform dust at rest with mass density ρ $[g/cm^3]$. At time $t = 0$ the plate is set in motion along its normal with initial Lorentz factor $\gamma(0) = \gamma_0$. The moving plate collides inelastically with the dust particles (which stick to its surface) and gradually decelerates.

   Find the evolution of its Lorentz factor $\gamma(t)$. If $\gamma_0 \gg 1$, at what time $\gamma(t) = \gamma_0/2$?

4. A one-dimensional non-relativistic particle interacts with the potential

$$V(x) = \lambda \frac{\hbar^2}{2m} \delta(x), \tag{3}$$

where λ is a constant and the prefactor is factored out to simplify the algebra.

(a) Calculate the reflection and transmission coefficients (probabilities) as a function of the incident particle wavenumber k.

(b) Calculate the scattering and bound states for $\lambda < 0$. Show that there is a single bound state, and that it is orthogonal to the scattering states.

5. Consider a hydrogen atom. The spin-orbit interaction at radius r is written as:

$$H_{\text{so}} = \frac{e^2}{2m^2c^2r^3}\vec{S} \cdot \vec{L}, \tag{4}$$

where $\vec{S}$ is the spin of the electron and $\vec{L}$.

(a) Describe in words the origin of the spin-orbit interaction.

(b) Construct the basis of wave functions that diagonalize H_{so}.

(c) Obtain the spin-orbit interaction energies for hydrogen in the state with radial quantum number n=2. You may express your answer in terms of the expectation values $\langle 1/r^3 \rangle$ of the hydrogen atom states (you do not need to calculate these expectation values explicitly).

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1. **Relativistic Harmonic Oscillator: Mueller**

   Consider the first relativistic correction, H_1, in the Hamiltonian for the one dimensional harmonic oscillator.

$$H = H_0 + H_1 = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2 - \frac{1}{2mc^2}\left(\frac{p^2}{2m}\right)^2 \tag{1}$$

   Evalulate the first order shift in the ground state energy due to H_1.
   Recall that the ground state wave function is given by

$$U_0(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \exp\left[-\frac{m\omega x^2}{2\hbar}\right] \tag{2}$$

   Solution: It is convenient to write

$$H_1 = -\frac{1}{2mc^2}\left(H_0 - \frac{1}{2}m\omega^2 x^2\right)^2 \tag{3}$$

   Then, the energy shift in the ground state is given by,

$$\begin{aligned}\Delta E &= \langle U_0|H_1|U_0\rangle \\ &= -\frac{1}{2mc^2}\int_{-\infty}^{\infty} dx \left(\frac{m\omega}{\pi\hbar}\right)^{1/2} \exp\left[-\frac{m\omega x^2}{\hbar}\right]\left(\underbrace{H_0}_{\langle U_0|H_0|U_0\rangle = E_0 = \hbar\omega/2} - \frac{1}{2}m\omega^2 x^2\right)^2 \end{aligned}\tag{4}$$

$$= -\frac{1}{2mc^2}\left(\frac{m\omega}{\pi\hbar}\right)^{1/2}\int_{-\infty}^{\infty} dx \exp\left[-\frac{m\omega x^2}{\hbar}\right]\left(\frac{\hbar\omega}{2}\right)^2\left(1 - \frac{m\omega x^2}{\hbar}\right)^2 \tag{5}$$

$$= -\frac{\hbar^2\omega^2}{8mc^2}\left(\frac{m\omega}{\pi\hbar}\right)^{1/2} 2\int_{0}^{\infty} dx \exp\left[-\frac{m\omega x^2}{\hbar}\right]\left(1 - \frac{m\omega x^2}{\hbar}\right)^2 \tag{6}$$

   Now switch variables to $y = m\omega x^2/\hbar$, $dy = (2m\omega/\hbar)x dx = 2(m\omega/\hbar)^{1/2}y^{1/2}$dx,

$$\Delta E = -\frac{\hbar^2\omega^2}{8\pi^{1/2}mc^2}\int_0^{\infty}\frac{dy}{y^{1/2}}\exp[-y]\left(1 - 2y + y^2\right) \tag{7}$$

$$= -\frac{\hbar^2\omega^2}{8\pi^{1/2}mc^2}\int_0^{\infty} dy \exp[-y]\left(y^{-1/2} - 2y^{1/2} + y^{3/2}\right) \tag{8}$$

$$= -\frac{\hbar^2\omega^2}{8\pi^{1/2}mc^2}\left[\Gamma(1/2) - 2\Gamma(3/2) + \Gamma(5/2)\right] = -\frac{\hbar^2\omega^2}{8\pi^{1/2}mc^2}\underbrace{\Gamma(1/2)}_{\sqrt{\pi}}\left[1 - 2\frac{1}{2} + \frac{3}{4}\right] \tag{9}$$

$$= -\hbar\omega\frac{3}{32}\frac{\hbar\omega}{mc^2} \tag{10}$$

2. **Relativity: Greene**

An asteroid is on a collision course with a space station located 5000 light-minutes from Earth. The asteroid is moving away from the Earth toward the space station at a speed of 3/5c along a trajectory which is a straight line connecting Earth and the space station. To save the station, NASA launches a missle from Earth at 4/5c. When the missile is launched, NASA determines that the asteroid is 400 light-minutes from Earth.

a) How many minutes should NASA set on a timer located on the missle so that it will explode just as it catches up to the asteroid? (In all parts of this problem, ignore subtleties having to do with acceleration.)

b) A few weeks later, another asteroid is on a similar collision course with the space station, travelling again at 3/5c. NASA decides to send another missile to destroy it, but this time they want the missile to pass the asteroid and explode only when according to sensors on the missile, the missile is 350 light minutes beyond the asteroid. In this scenario, how many minutes should NASA set on the missile's timer?

c) From the perspective of the missile's frame of reference, how far away is the asteroid when the missile is launched?

Solution.

Work in units with $c = 1$ light-minute/minute.

a) The time to catch asteroid Δt on the Earth clock is given by $\frac{4}{5}\Delta t = 400 + \frac{3}{5}\Delta t$ and so $\Delta t = 2000$. Because the missile is moving, its clock ticks off slow due to time dilation, and so the engineers set it to $2000/\gamma = 2000/(\frac{5}{3}) = 1200$ minutes, where $\gamma = (1 - [4/5]^2)^{-1/2} = 5/3$.

b) Relative to the missile, the asteroid has a speed of $\beta = (\frac{4}{5} - \frac{3}{5})/(1 - \frac{4}{5}\frac{3}{5}) = \frac{5}{13}$. Thus, for the missile to travel 350 light-minutes beyond the asteroid, from its perspective, it will take an additional $350/\frac{5}{13} = 910$ minutes, beyond the 1200 minutes required to reach the asteroid, for a total of 2110 minutes.

c) Consider the primed frame to be that of the missile, which is moving with $\beta = 4/5$ ($\gamma = 5/3$) with respect to the lab frame. Using the Lorentz transformation, the asteroid's trajectory in the Earth's frame $(t, 400 + \frac{3}{5}t)$ intersects the x' axis ($t' = 0$) for the missile at time $t = \beta(400 + (3/5)t)$, or $t = 8000/13$, i.e. when the missile is at $(t, \frac{4}{5}t) = \frac{80}{13}(100, 125)$. In the missile's coordinates, this location is $(t' = 0, x' = 400\frac{15}{13})$, and hence from the missile's perspective, the asteroid is at a distance of $400\frac{15}{13}$ light-minutes when it launches.

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3. **Relativity: Beloborodov** To the typesetter: in the text below please replace "inelastically with the dust particles" with "inelastically with the dust particles (which stick to its surface)"

Relativity:

A thin plate with surface rest-mass density Σ_0 [g/cm^2] is surrounded by uniform dust at rest with mass density ρ [g/cm^3]. At time $t = 0$ the plate is set in motion along its normal with initial Lorentz factor $\gamma(0) = \gamma_0$. The moving plate collides inelastically with the dust particles and gradually decelerates. Find the evolution of its Lorentz factor $\gamma(t)$. If $\gamma_0 \gg 1$, at what time $\gamma(t) = \gamma_0/2$?

Solution:

Let $x(t)$ be the displacement of the plate; its velocity is $v = dx/dt$. The column density of the swept-up dust is $\Sigma_d(t) = \rho\, x(t)$. Conservation of energy and momentum gives

$$\gamma\Sigma = \gamma_0\Sigma_0 + \Sigma_d, \tag{1}$$

$$v\gamma\Sigma = v_0\gamma_0\Sigma_0. \tag{2}$$

Note that $\Sigma > \Sigma_0 + \Sigma_d$, as the growing inertial mass of the plate includes heat produced in the inelastic collisions. To get rid of the unknown Σ divide the two equations; then one finds

$$v = \frac{v_0\gamma_0\Sigma_0}{\gamma_0\Sigma_0 + \Sigma_d} \qquad \text{or} \qquad \frac{dx}{dt} = \frac{v_0\gamma_0\Sigma_0}{\gamma_0\Sigma_0 + \rho x}. \tag{3}$$

This equation can be solved for $x(t)$. Introducing $x_\star \equiv \gamma_0\Sigma_0/\rho$, one obtains

$$x + \frac{x^2}{2x_\star} = v_0\, t \qquad \Rightarrow \qquad x(t) = x_\star\left(\sqrt{1 + \frac{2v_0 t}{x_\star}} - 1\right), \tag{4}$$

and hence

$$v(t) = v_0\left(1 + \frac{2t}{t_\star}\right)^{-1/2}, \qquad \gamma(t) = \gamma_0\left(\frac{1 + 2t/t_\star}{1 + 2\gamma_0^2\, t/t_\star}\right)^{1/2}, \tag{5}$$

where $t_\star = x_\star/v_0$. $\gamma = \gamma_0/2$ corresponds to $t = 3t_\star/(2\gamma_0^2 - 8)$, which gives $t \approx 3\Sigma_0/2\gamma_0\rho c$ when $\gamma_0 \gg 1$.

General:

Electric current I is circulating along an ideally conducting thin hoop of mass M and radius R. The hoop is placed in infinite vacuum space and set in rotation with angular velocity $\vec{\Omega}_0$ that is perpendicular to the hoop axis. How will its angular velocity evolve with time?

Solution:

The magnetic dipole moment of the hoop $\mu = \pi R^2 I/c$ rotates about $\vec{\Omega}$ and hence its second time derivative is $|\ddot{\vec{\mu}}| = \Omega^2\mu$. This generates magneto-dipole radiation with power $\dot{E} = 2\ddot{\vec{\mu}}^2/3c^3$, and hence the hoop kinetic energy $E = MR^2\Omega^2/4$ is gradually decreasing,

$$\frac{d}{dt}\left.\frac{MR^2\Omega^2}{4}\right) = -\frac{2}{3c^3}\left.\frac{\Omega^2\pi R^2 I}{c}\right)^2 \qquad \Rightarrow \qquad \Omega(t) = \left.\frac{1}{\Omega_0^2} + \frac{8\pi^2 R^2 I^2\, t}{3Mc^5}\right)^{-2}.$$

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4. **Scattering/Bound States - Metzger**

A one-dimensional non-relativistic particle interacts with the potential

$$V(x) = \lambda \frac{\hbar^2}{2m} \delta(x), \tag{11}$$

where λ is a constant and the prefactor is factored out to simplify the algebra.
(a) Calculate the reflection and transmission coefficients (probabilities) as a function of the incident particle wavenumber k.
(b) Calculate the scattering and bound states for $\lambda < 0$. Show that there is a single bound state, and that it is orthogonal to the scattering states.

Solution.
(a) The Schrodinger equation reads

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V(x)\psi = E\psi \Rightarrow -\frac{d^2}{dx^2} + \lambda\delta(x)\psi(x) = k^2\psi(x) \tag{12}$$

The solution is

$$\psi(x) = e^{ikx} + Re^{-ikx}, x < 0 \tag{13}$$

$$\psi(x) = Te^{ikx}, x > 0, \tag{14}$$

where $E = \hbar^2 k^2/2m$ is the incident energy.
Boundary conditions: (1) $\psi(x)$ is continuous at $x = 0$,

$$\Rightarrow 1 + R = T \tag{15}$$

while (2) integrating the Schrodinger equation across $x = 0$ gives

$$\frac{d\psi}{dx}|_{x-\epsilon} - \frac{d\psi}{dx}|_{x+\epsilon} + \lambda\psi(0) = 0 \tag{16}$$

$$\Rightarrow ik(1 - R) + \lambda(1 + R) = ikT \tag{17}$$

Solving for the amplitudes,

$$R = \frac{-i\lambda}{2k + i\lambda}, T = \frac{2k}{2k + i\lambda} \tag{18}$$

and the resulting transmission probabilities

$$|R|^2 = \frac{\lambda^2}{4k^2 + \lambda^2}, \quad |T|^2 = \frac{4k^2}{4k^2 + \lambda^2} \tag{19}$$

(b) The scattering states have already been calculated since part (a) did not depend on the sign of λ. The bound states have wave functions which are exponentially damped,

$$\psi_B(x) = Ae^{\kappa x}, x < 0, \tag{20}$$

$$\psi_B(x) = Be^{-\kappa x}, x > 0, \tag{21}$$

where the energy is $E = -\hbar^2\kappa^2/2m$. The same boundary conditions as before lead to

$$A = B, \quad \kappa A + \lambda A = -\kappa B \tag{22}$$

with solution

$$2\kappa = -\lambda \tag{23}$$

This is only a sensible solution for $\kappa > 0$, i.e. for $\lambda < 0$. $A = B$ are fixed by normalizing the bound state,

$$_B(x) = \sqrt{\kappa}e^{-\kappa|x|} \tag{24}$$

The bound state is unique, since there is a single solution.
The overlap between the bound state ψ_B with a scattering state ψ_k of wavenumber k:

$$\int_{-\infty}^{\infty} dx\psi_B(x)^*\psi_k(x) = \int_{-\infty}^{0} \sqrt{\kappa}e^{\kappa x}(1+Re^{-ikx})+\int_{0}^{\infty} \sqrt{\kappa}e^{-\kappa x}(Te^{-ikx}) = \sqrt{\kappa}\left[\frac{R-T}{\kappa - ik} + \frac{1}{\kappa}\right] = 0 \tag{25}$$

where in the final equality we have used equation (17)

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5. **Spin-Orbit Interaction in Hydrogen Atom - Pinczuk**
Consider a hydrogen atom. The spin-orbit interaction at radius r is written as:

$$H_{\text{so}} = \frac{e^2}{2m^2c^2r^3}\vec{S}\cdot\vec{L}, \tag{26}$$

where $\vec{S}$ is the spin of the electron and $\vec{L}$.
(a) Describe in words the origin of the spin-orbit interaction.
(b) Construct the basis of wave functions that diagonalize H_{so}.
(c) Obtain the spin-orbit interaction energies for hydrogen in the state with radial quantum number n=2. You may express your answer in terms of the expectation values $\langle 1/r^3\rangle$ of the hydrogen atom states (you do not need to calculate these expectation values explicitly).
Solution.
(a) In the rest frame of the electron the proton is moving with velocity $\vec{v}$, that produces a magnetic field $\vec{B}\propto\vec{L}$. The coupling of this field to the intrinsic magnetic moment from $\vec{S}$ is the spin-orbit interaction H_{so}. In fact, this simple-minded picture only accounts for half of the effect described by equation (26). The electron's motion in the proton's electric field alone will give a spin-orbit Hamiltonian

$$H_{so} = \frac{g_e e^2}{2m^2c^2r^3}\vec{S}\cdot\vec{L} \tag{27}$$

which is $g_e \sim 2$ times larger than that in eq. (26). However, equation (27) actually describes the motion of the spin relative to a reference system that, because of its circular acceleration, is actually rotating in the laboratory. The Hamiltonian that describes both effects has a factor $(g_e - 1) \sim 1$, which is the correct expression given in equation (26).
(b) Rewrite the spin-orbit interaction as

$$H_{so} = \frac{e^2}{2m^2c^2r^3}\vec{S}\cdot\vec{L} = \frac{e^2}{4m^2c^2r^3}(J^2 - L^2 - S^2), \tag{28}$$

where $\vec{J}$ is the total angular momentum. The states with quantum numbers J, L, and S make this Hamiltonian diagonal. These states can be written as $|j, m, l, s\rangle$, where m are the quantum number associated with the z components of the total angular momentum.
In this basis the matrix elements of H_{so} are given by

$$\langle H_{\text{so}}\rangle = \frac{\hbar^2 e^2}{2m^2c^2r^3}\langle r^{-3}\rangle_{nl}\left[j(j+1) - l(l+1) - \underbrace{s(s+1)}_{=\frac{3}{4}}\right] \tag{29}$$

(c) For $n = 2$ the allowed states of l are 0 or 1. The allowed values of j are $1 \pm \frac{1}{2}$.
The value $j = 1/2$ is obtained from $l = 0, 1$. There are two terms.
The value $j = 3/2$ is obtained from $l = 1$ only (one term here).
There are also two matrix elements:

$$\langle r^{-3} \rangle_{2,0} \equiv A_0 \tag{30}$$

and

$$\langle r^{-3} \rangle_{2,1} \equiv A_1 \tag{31}$$

The spin-orbit interaction energies are obtained with equations (29) and (31).