Columbia University
Department of Physics
QUALIFYING EXAMINATION

Monday, January 11, 2016
3:10PM to 5:10PM
Classical Physics
Section 2. Electricity, Magnetism & Electrodynamics

Two hours are permitted for the completion of this section of the examination. Choose 4 problems out of the 5 included in this section. (You will not earn extra credit by doing an additional problem). Apportion your time carefully.

Use separate answer booklet(s) for each question. Clearly mark on the answer booklet(s) which question you are answering (e.g., Section 2 (Electricity etc.), Question 2, etc.).

Do **NOT** write your name on your answer booklets. Instead, clearly indicate your **Exam Letter Code**.

You may refer to the single handwritten note sheet on $8\frac{1}{2}$" $\times$ 11" paper (double-sided) you have prepared on Classical Physics. The note sheet cannot leave the exam room once the exam has begun. This note sheet must be handed in at the end of today's exam. Please include your Exam Letter Code on your note sheet. No other extraneous papers or books are permitted.

Simple calculators are permitted. However, the use of calculators for storing and/or recovering formulae or constants is NOT permitted.

Questions should be directed to the proctor.

Good Luck!

1. A current I flows down a long straight wire of radius a. Assume the wire is made of linear material with magnetic susceptibility χ_m, and the current is distributed uniformly across the cross section of the wire.

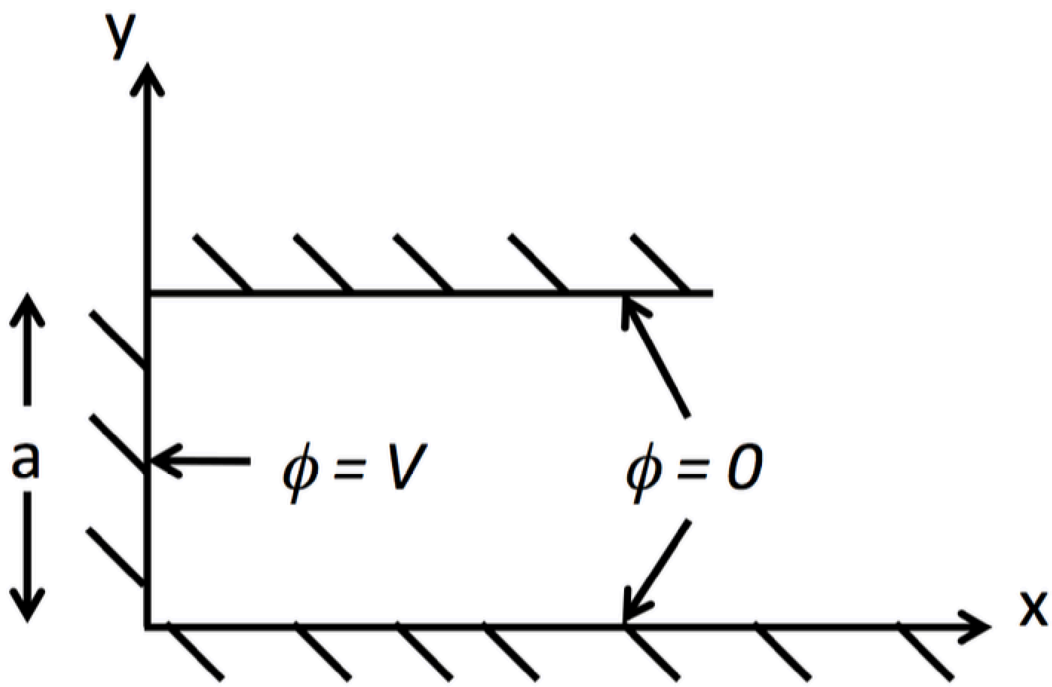
   (a) Calculate the magnetic field a distance s from the axis. Consider separately the regions both inside the wire ($s < a$) and outside ($s > a$).

   (b) Calculate all of the bound currents in the problem. What is the net bound current flowing down the wire?

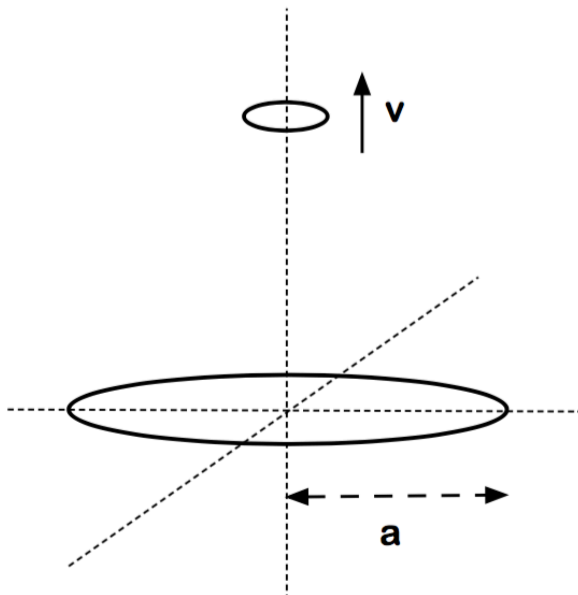
2. An infinitely long 2-D slot has a width a. The walls are conductors held at fixed potentials $\phi = 0$ and $\phi = V$, as shown.

   (a) Determine the electric potential at an arbitrary location (x, y) inside the slot.

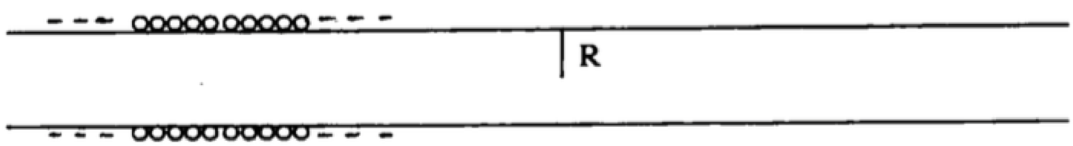
   (b) A positron is released from rest at the coordinates (x,y) = (a, a/2). Find an expression for the force on the positron. Where will it be located at time $t \to \infty$?



3. A thin, circular conducting ring of radius a lies fixed in the $x - y$ plane centered on the z axis. It is driven by a power supply such that it carries a constant current I. Another thin conducting ring of radius b, with $b << a$, and resistance R is centered on and is normal to the z axis. This second ring is moved along the z axis at constant velocity v such that it's center is located at $z = vt$. Estimate, using what ever approximations you consider appropriate, the following quantities including the full time dependence.
   (a) The current in the moving ring.
   (b) The force required to keep the ring moving at constant velocity.



4. A long cylindrical solenoid of radius R and length $L \gg R$ is tightly wound with a single layer of wire (see below). The number of turns per unit length is N/L. The wire breaks when the tension in the wire is greater than T. Find the maximum current that can be carried by the wire.

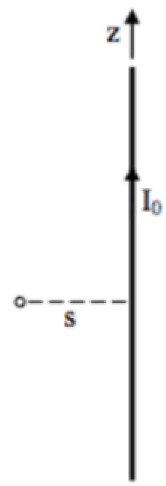


N/L turns/length

5. The general expressions for the scalar and vector potentials are

$$V(\vec{r},t) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\vec{r}',t_r)d^3r'}{|\vec{r}-\vec{r}'|}, \quad \vec{A}(\vec{r},t) = \frac{\mu_0}{4\pi}\int \frac{J(\vec{r}',t_r)d^3r'}{|\vec{r}-\vec{r}'|}, \tag{1}$$

where t_r is the retarded time.



A long (effectively infinite) neutral wire on the z-axis has zero current for $t < 0$. At $t = 0$ a steady current I_0 is suddenly turned on in the $+\hat{z}$ direction (see Figure).

(a) Consider a point at a distance s from the wire ($z = 0$). At what time do the electric and/or magnetic fields first become non-zero at this point? Hereafter call this time t_s ('s' for when the field starts at position s).

(b) What is the value of the scalar potential V at position s at time $t > t_s$?

(c) What is the direction of the vector potential $\vec{A}$ at position s at time $t > t_s$?

(d) What is the direction of the electric field $\vec{E}$ at position s at time $t > t_s$?

(e) What is the direction of the magnetic field $\vec{B}$ at position s at time $t > t_s$?

(f) Write an integral expression for the magnitude of the vector potential $A(s,t)$ at times $t > t_s$.

UNI Here:

1. **Current Carrying Wire: Brooijmans**

A current I flows down a long straight wire of radius a. Assume the wire is made of linear material with magnetic susceptibility χ_m, and the current is distributed uniformly across the cross section of the wire.

a) Calculate the magnetic field a distance s from the axis. Consider separately the regions both inside the wire ($s < a$) and outside ($s > a$).
b) Calculate all of the bound currents in the problem. What is the net bound current flowing down the wire?

Solution

a) Ampere's Law gives, for $s < a$

$$\oint \overrightarrow{H}.\overrightarrow{dl} = H2\pi s = I_{f,enc} = I\frac{s^2}{a^2}, \tag{1}$$

and, for $s > a$,

$$\oint \overrightarrow{H}.\overrightarrow{dl} = H2\pi s = I_{f,enc} = I. \tag{2}$$

So for $s < a$, where $\mu = \mu_0(1 + \chi_m)$,

$$\overrightarrow{H} = \frac{Is}{2\pi a^2}\hat{\phi} \text{ and } \overrightarrow{B} = \mu H = \frac{\mu_0(1+\chi_m)Is}{2\pi a^2}\hat{\phi}, \tag{3}$$

and for $s > a$, where $\mu = \mu_0$,

$$\overrightarrow{H} = \frac{I}{2\pi s}\hat{\phi} \text{ and } \overrightarrow{B} = \mu H = \frac{\mu_0 I}{2\pi s}\hat{\phi}. \tag{4}$$

b) The volumetric bound current is given by,

$$\overrightarrow{J_b} = \chi_m \overrightarrow{J_f} = \frac{\chi_m I}{\pi a^2}\hat{z}, \tag{5}$$

The bound surface current is given by

$$\overrightarrow{K_b} = \overrightarrow{M} \times \hat{n} = \chi_m \overrightarrow{H} \times \hat{n} = \frac{\chi_m I}{2\pi a}(-\hat{z}) \tag{6}$$

Therefore the total bound current down the wire (volumetric + surface) is given by

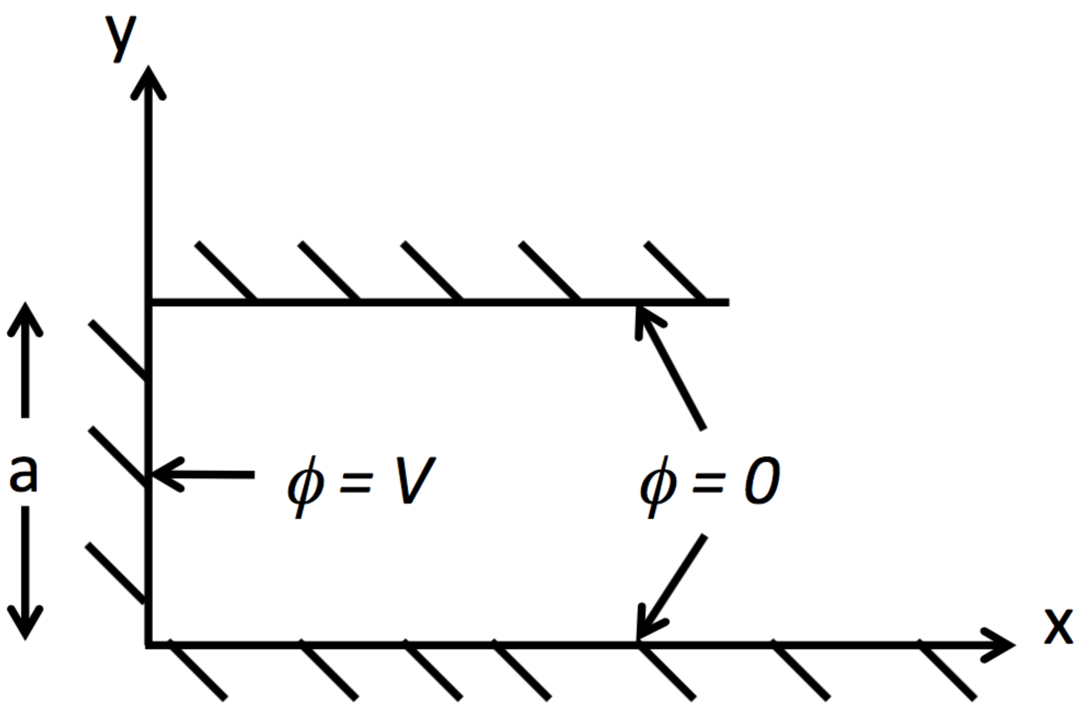
$$\overrightarrow{I_b} = \overrightarrow{J_b}\pi a^2 + \overrightarrow{K_b}2\pi a = 0. \tag{7}$$

UNI Here:

2. **Conducing Walls: Humensky**

An infinitely long 2-D slot has a width a. The walls are conductors held at fixed potentials $\phi = 0$ and $\phi = V$, as shown.

a) Determine the electric potential at an arbitrary location (x, y) inside the slot.
b) A positron is released from rest at the coordinates (x,y) = (a, a/2). Find an expression for the force on the positron. Where will it be located at time $t \to \infty$?



Solution.
(a) Solve Laplace equation inside the slot,

$$\nabla^2 \phi = 0, \tag{8}$$

with boundary conditions $\phi(x, y = 0) = \phi(x, y = a) = 0$, $\phi(0, y) = V$, and $\underset{x\to\infty}{lim}\,\phi = 0$. Using separation of variables $\phi(x, y) = X(x)Y(y)$, $\nabla^2\phi = 0$ becomes

$$\frac{d^2X/dx^2}{X} + \frac{dY^2/dy^2}{Y} = 0 \Rightarrow \frac{d^2X/dx^2}{X} = K, \frac{d^2Y/dy^2}{Y} = -K, \tag{9}$$

where K is a constant. Choose $K = k^2$ positive to match periodic boundary conditions in y, where k is another positive constant. The solutions then become

$$X = e^{\pm kx}, Y = \sin(ky), \cos(ky) \tag{10}$$

The boundary condition $\phi(x, y = 0) = \phi(x, y = a) = 0$ demands we pick

$$Y_n = \sin(k_n y), \tag{11}$$

where $k_n = n\pi/a, n = 0, 1, 2, 3, ...$ is required to match the boundary conditions. The boundary condition $\underset{x\to\infty}{lim}\, \phi = 0$ requires we discard the exponentially growing solution $X_n = e^{+k_n x}$, such that the general solution becomes

$$\phi(x, y) = \sum_n A_n e^{-n\pi x/a} \sin(\pi n y/a). \tag{12}$$

The constants A_n are fixed by the $\phi(0, y) = V$ boundary condition:

$$\phi(0, y) = V = \sum_n A_n \sin(n\pi y/a) \tag{13}$$

Multiplying by the orthogonal function $\sin(m\pi y/a)$ and integrating over y gives

$$A_n = \frac{2}{a}\int_0^a V \sin(n\pi y/a)dy = -\frac{2V}{n\pi}\cos(n\pi y/a)|_0^a = \frac{4V}{\pi n} \,\mathrm{if\, n\, odd}, = 0 \,\mathrm{if\, n\, even} \tag{14}$$

Thus,

$$\phi(x, y) = \frac{4V}{\pi}\sum_{odd\, n} \frac{1}{n} e^{-n\pi x/a} \sin(n\pi y/a) \tag{15}$$

(b) The force on the positron is given by

$$\vec{F} = e\vec{E} = -e\nabla\phi \tag{16}$$

$$F_x = -e\frac{\partial \phi}{\partial x} = \frac{4Ve}{a}\sum_{odd\, n} e^{-n\pi x/a} \sin(n\pi y/a) \tag{17}$$

$$F_y = -e\frac{\partial \phi}{\partial y} = \frac{-4Ve}{a}\sum_{odd\, n} e^{-n\pi x/a} \cos(n\pi y/a) \tag{18}$$

At (x,y) = (a,a/2) we have

$$F_x = \frac{4Ve}{a}\sum_{odd\, n} e^{-n\pi} \sin(n\pi/2) \tag{19}$$

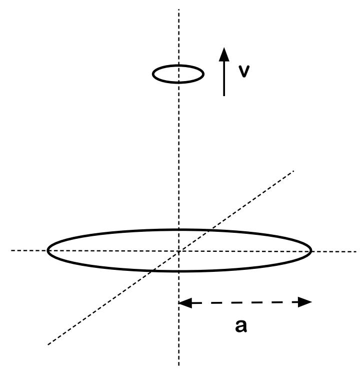
$$F_y = \frac{-4Ve}{a}\sum_{odd\, n} e^{-n\pi} \cos(n\pi/2) = 0 \tag{20}$$

At large x the $n = 1$ will dominate the sum in F_x, such that

$$\vec{F} \Rightarrow \frac{4Vd}{a} e^{-\pi}\hat{x} \tag{21}$$

and hence as $t \to \infty$ the positron x coordinate will move to $+\infty$.

3. **Conducting Ring: Cole**



A thin, circular conducting ring of radius a lies fixed in the $x-y$ plane centered on the z axis. It is driven by a power supply such that it carries a constant current I. Another thin conducting ring of radius b, with $b << a$, and resistance R is centered on and is normal to the z axis. This second ring is moved along the z axis at constant velocity v such that it's center is located at $z = vt$. Estimate, using what ever approximations you consider appropriate, the following quantities including the full time dependence.

(a) The current in the moving ring.

(b) The force required to keep the ring moving at constant velocity.

Solution. a) Near the z axis the magnetic field points primarily in the z direction though it has a small radial component which gives zero contribution to the magnetic flux through the moving ring. The z component of the field varies slowly with r near the z axis, so we can approximate the magnetic flux through the moving loop as

$$\Phi_m = \pi b^2 B(r = 0, z) \tag{22}$$

A straight-forward application of the Biot-Savart law gives the magnetic field on the z axis a distance z from the fixed ring (SI units)

$$B_z(r = 0, z) = \frac{\mu_0 a^2 I}{2\left(z^2 + a^2\right)^{3/2}} \tag{23}$$

The EMF resulting from the motion of the second ring is given by

$$\epsilon = \frac{d\Phi_m}{dt} = v\frac{d\Phi_m}{dz} = -\frac{3\mu_0 \pi b^2 a^2 I v z}{2\left(z^2 + a^2\right)^{5/2}} \tag{24}$$

The induced current is then (substituting $z = vt$)

$$i = \frac{\epsilon}{R} = -\frac{3\mu_0 \pi b^2 a^2 I v^2 t}{2R\left(v^2t^2 + a^2\right)^{5/2}} \tag{25}$$

b) The z component of the magnetic field produces a radially-directed force on the current moving in the ring. However, the small radial component of the magnetic field produces a force in the z direction that must be counteracted in order to keep the ring moving at constant velocity. The radial component can be evaluated in many different ways. Here, we "boost strap" it using $\nabla \cdot \vec{B} = 0$ and a taylor expansion of the x and y components of $\vec{B}$ in the x-y plane:

$$\vec{B}_x(x, 0, z) \approx x \frac{\partial B_x}{\partial x}|_{0,0,z} \tag{26}$$

$$\vec{B}_y(0, y, z) \approx y \frac{\partial B_y}{\partial y}|_{0,0,z} \tag{27}$$

Symmetry requires that

$$\frac{\partial B_x}{\partial x}|_{0,0,z} = \frac{\partial B_y}{\partial y}|_{0,0,z}, \tag{28}$$

such that applying $\nabla \cdot \vec{B} = 0$ at (0,0,0) results in

$$\frac{\partial B_x}{\partial x}|_{0,0,z} = -\frac{1}{2}\frac{\partial B_z}{\partial z}|_{0,0,z} = \frac{3\mu_0 a^2 I z}{4(z^2 + a^2)^{5/2}} \tag{29}$$

Inserting this result into the Taylor expansion and nothing that cylindrical symmetry allows us to gernalize $x \to r$, we conclude

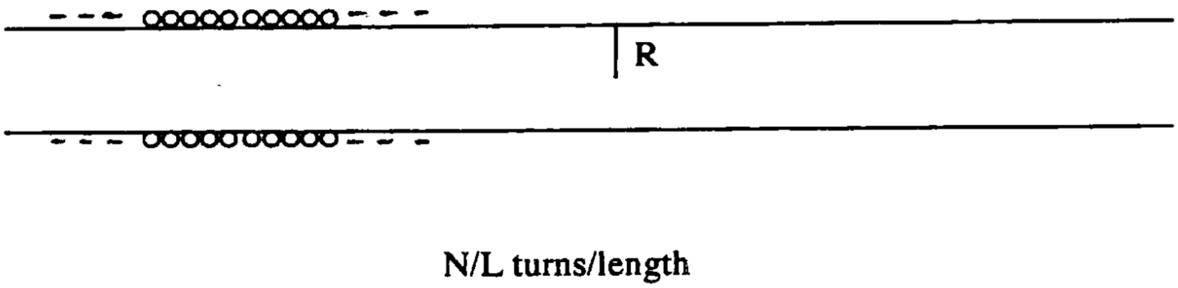
$$\vec{B}_r(r, z) \approx \frac{3\mu_0 a^2 I z r}{4(z^2 + a^2)^{5/2}} \tag{30}$$

The radial field produces a total force $-2\pi b i B_r(b, z)\hat{k}$. Thus, a force $\vec{F} = 2\pi b i B_r(b, z)\hat{k}$ has to be applied to keep the ring moving at constant velocity. Putting everything together,

$$F = \frac{9}{4}\frac{\mu_0^2 \pi^2 a^4 b^4 I^2 v^3 t^2}{R(v^2t^2 + a^2)^5} \tag{31}$$

UNI Here:

4. **Solenoid: Metzger** A long cylindrical solenoid of radius R and length $L \gg R$ is tightly wound with a single layer of wire (see below). The number of turns per unit length is N/L. The wire breaks when the tension in the wire is greater than T. Find the maximum current that can be carried by the wire.



Solution
The magnetic field in the solenoid is given by

$$B = \mu_0 (N/L) I \tag{32}$$

From the Maxwell stress tensor, the transverse pressure (radial force per area) on the coil is $B^2/2\mu_0$ (the other components of the stress vanish by symmetry).
The net force on the top half of the wire is therefore given by

$$F_y = L \int_0^{\pi} R d\theta \frac{B^2}{2\mu_0} \sin\theta = LR\frac{B^2}{2\mu_0}[-\cos\theta]_0^{\pi} = LR\frac{B^2}{\mu_0} \tag{33}$$

The tension in each wire is thus given by

$$T = \frac{F_y}{2N} = \frac{LR}{2N}\frac{B^2}{\mu_0} = \frac{LR}{2N\mu_0}\left[\frac{\mu_0 N I}{L}\right]^2 = \frac{\mu_0 N R I^2}{2L}, \tag{34}$$

where the factor of two in the denominator results because the force per turn is shared between the two sides of the wire. The maximum current is thus given by

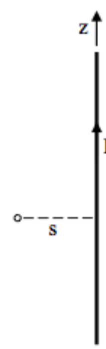
$$I_{\max} = \left(\frac{2LT}{\mu_0 N R}\right)^{1/2} \tag{35}$$

UNI Here:

5. **Retarded Potentials: Metzger**
The general expressions for the scalar and vector potentials are

$$V(\vec{r},t) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\vec{r}',t_r)d^3r'}{|\vec{r}-\vec{r}'|}, \quad \vec{A}(\vec{r},t) = \frac{\mu_0}{4\pi}\int \frac{J(\vec{r}',t_r)d^3r'}{|\vec{r}-\vec{r}'|}, \tag{36}$$

where t_r is the retarded time.



A long (effectively infinite) neutral wire on the z-axis has zero current for $t < 0$. At $t = 0$ a steady current I_0 is suddenly turned on in the $+\hat{z}$ direction (see Figure).

(a) Consider a point at a distance s from the wire ($z = 0$). At what time do the electric and/or magnetic fields first become non-zero at this point? Hereafter call this time t_s ('s' for when the field starts at position s).

(b) What is the value of the scalar potential V at position s at time $t > t_s$?

(c) What is the direction of the vector potential $\vec{A}$ at position s at time $t > t_s$?

(d) What is the direction of the electric field $\vec{E}$ at position s at time $t > t_s$?

(e) What is the direction of the magnetic field $\vec{B}$ at position s at time $t > t_s$?

(f) Write an integral expression for the magnitude of the vector potential $A(s,t)$ at times $t > t_s$.

Solution.
(a) retarded time $\Rightarrow$ 'signal' that current has turned reaches distance s on the light travel time $t_s = s/c$
(b) $V = 0$ because the wire is neutral $\Rightarrow \rho = 0 \Rightarrow V = 0$
(c) $\vec{A}$ points in the $\hat{z}$ direction, the same direction as $\vec{J}$.
(d)

$$\vec{E} = -\nabla V - \frac{\partial \vec{A}}{\partial t} \underset{V=0}{=} -\frac{\partial \vec{A}}{\partial t} \tag{37}$$

since $\vec{A}$ is in the $\hat{z}$ direction and increasing, $\vec{E} = -\partial\vec{A}/\partial t$ is in the -$\hat{z}$ direction.
(e) $\vec{B} = \nabla \times \vec{A}$ points out of the page. One way to see this is to note that the Poynting Vector $\vec{s} = \frac{1}{\mu_0}\vec{E} \times \vec{B}$ must point away from the wire since EM energy is flowing outwards to fill space.
(f) At times $t > t_s$, the magnitude of the vector potential can be written in this form:

$$A(s,t) = \frac{\mu_0}{4\pi} \int_{-\sqrt{(ct)^2 - s^2}}^{+\sqrt{(ct)^2 - s^2}} \frac{I_0}{\sqrt{s^2 + z^2}} dz \tag{38}$$

The limits of integration represent the part of the wire in causal contact with s,

$$z^2 + s^2 = (ct)^2 \Rightarrow z = \pm\sqrt{(ct)^2 - s^2} \tag{39}$$