Columbia University
Department of Physics
QUALIFYING EXAMINATION

Wednesday, January 11, 2012
3:10PM to 5:10PM
Modern Physics
Section 4. Relativity and Applied Quantum Mechanics

Two hours are permitted for the completion of this section of the examination. Choose 4 problems out of the 5 included in this section. (You will not earn extra credit by doing an additional problem). Apportion your time carefully.

Use separate answer booklet(s) for each question. Clearly mark on the answer booklet(s) which question you are answering (e.g., Section 4 (Relativity and Applied QM), Question 2, etc.).

Do **NOT** write your name on your answer booklets. Instead, clearly indicate your **Exam Letter Code**.

You may refer to the single handwritten note sheet on $8\frac{1}{2}$" $\times$ 11" paper (double-sided) you have prepared on Modern Physics. The note sheet cannot leave the exam room once the exam has begun. This note sheet must be handed in at the end of today's exam. Please include your Exam Letter Code on your note sheet. No other extraneous papers or books are permitted.

Simple calculators are permitted. However, the use of calculators for storing and/or recovering formulae or constants is NOT permitted.

Questions should be directed to the proctor.

Good Luck!

1. Consider an isolated spin-1 ion in a crystal. Due to a particular crystal field, the spin Hamiltonian can be described by

$$\tilde{H} = AJ_z^2 + \Delta\left(J_x^2 - J_y^2\right)$$

where $\vec{J}$ is the spin-1 operator. Here, A and Δ are constants and $A \gg \Delta > 0$.

(a) Ignoring small terms in the xy-plane (*i.e.*, set $\Delta = 0$), find the eigenvalues and eigenvectors in spin-1 space. Show that some eigenstates are degenerate.

(b) Now consider a finite but small Δ, such that we can use the second term in the Hamiltonian as the perturbation term. With this perturbation, the degeneracy in (a) is lifted. Find out the eigenvalues and eigenvectors up to first order in Δ.

Hint: The lowering and raising operators are defined as

$$J_{\pm} = J_x + iJ_y$$

$$J_{\pm}\left|j; m_z\right\rangle = \hbar\sqrt{j(j+1) - m_z(m_z \pm 1)}\left|j; m_z\right\rangle$$

2. Tritium, ^{3}H, is highly radioactive and decays with a half-life of 12.3 years to ^{3}He by the emission of an electron and an electron anti-neutrino from its nucleus. The electron's average kinetic energy is 5.7 keV. Explain why its departure can be treated as sudden in the sense that the electron of the original tritium atom barely moves while the ejected electron leaves.

   Calculate the probability that the newly-formed ^{3}He atom is in an excited state, by evaluating $\langle 1, 0, 0; Z = 2|1, 0, 0; Z = 1\rangle$. Here the notation gives the n, l, m values for the electronic state.

3. Consider a hydrogen atom. The spin-orbit interaction is written as:

$$H_{\text{so}} = A\vec{S} \cdot \vec{L},$$

where $\vec{S}$ is the spin of the electron, $\vec{L}$ is the orbital angular momentum, and $A = (e^2/2m^2c^2r^3)$.

   (a) Describe in words the origin of the spin-orbit interaction.

   (b) Construct the basis of wave functions that diagonalize H_{so}.

   (c) Obtain the spin-orbit interaction energies for hydrogen in the state with radial quantum number $n = 2$.

4. A hypothetical particle has negative squared mass such that we can write $m = i\mu$ where μ is a real quantity with units of mass. In all other respects, the particle satisfies the rules of special relativity, though sometimes in surprising ways. In answering the questions below you are encouraged to use natural units.

   (a) Find expressions for the energy, E, and momentum magnitude, p, of the particle in terms of its velocity β.

   (b) What are the maximum and minimum velocities for the particle and at what values of E and p are the maximum and minimum velocities attained?

   (c) Suppose a measurement of the velocity of the hypothetical particle yields $\beta - 1 \approx 2 \times 10^{-5}$ for $E \approx 30$ GeV. Estimate the value of μ that would be consistent with the measurement.

   (d) Consider the free propagation of our hypothetical particle in the "lab" frame at some velocity β. Suppose we define two events along the spacetime trajectory of the particle separated by a time Δt as observed in the lab frame. Show that it is possible to find another Lorentz frame in which the particle appears to be propagating backward in time. Evaluate the kinematics of the particle in that frame. How might an observer in that frame understand these results?

5. In this problem, please measure energies in MeV (mega-electron volts), velocities in units of the speed of light c and rest mass in units of MeV/c^2. Consider a collision between two particles, each of rest mass $m = 0.5MeV/c^2$. In this problem you are asked to compare two reference frames: the center of mass reference frame and the 'lab' reference frame in which one of the two particles is not moving. Suppose that in the center of mass reference frame the two particles are initially moving along x with velocities $v = \pm 0.8c$. In the lab frame one particle has velocity $v = 0$ and the other particle has a velocity which is directed along negative x.

   (a) Suppose that the collision is elastic, and that after the collision the two particles emerge moving along the $\pm y$ direction in the center of mass frame. Please give the angles made by the trajectories of the two particles with respect to the x axis, in the lab frame.

   (b) Suppose now that what collides are a particle and its antiparticle, so that the two colliding particles annihilate and produce two photons which, in the center of mass frame, move off in the $\pm y$ direction. Please find the momenta of the two photons, in the lab frame.

Applied Quantum: Spin-1 in a crystal field

Consider an isolated spin-1 ion in a crystal. Due to a particular crystal field, the spin Hamiltonian is now described by

$$\tilde{H} = AJ_z^2 + \Delta(J_x^2 - J_y^2)$$

where $\vec{J}$ is spin-1 operator. Here, A and Δ are constants and $A >> \Delta > 0$.

(a) Ignoring small term in x-y plan (i.e., set $\Delta = 0$), find out the eigen values and eigen vectors in spin-1 space. Show that some eigen states have degeneracy.

(b) Now consider the finite but small E, such that we can use the second term in the Hamiltonian as the perturbation term. With this perturbation, the degeneracy in (a) is lifted. Find out the eigen values and eigen vectors up to the first order in Δ.

[Hint: The lowering and raising operators are defined as

$$J_\pm = J_x + i\,J_y$$

$$J_\pm \left| j; m_z \right\rangle = \hbar\sqrt{j(j+1) - m_z(m_z \pm 1)}\,\left| j; m_z \right\rangle$$]

Spin-1 in a crystal field

(a) For spin-1, using $\{|j=1, m_z=1\rangle, |1,0\rangle, |1,-1\rangle\}$ as basis

$$J_z = \hbar \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$\check{H} = A J_z^2 = A\hbar^2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Ground state $|j=1, m=0\rangle$, $E=0$

Excited state $|j=1, m=\pm1\rangle$, $E = A\hbar^2$ doubly degenerate

(b) $J_\pm |j=1, m\rangle = \hbar\sqrt{1(1+1) - m(m\pm1)}$

$$\Rightarrow \quad J_+ = \hbar \begin{pmatrix} 0 & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{2} \\ 0 & 0 & 0 \end{pmatrix}, \quad J_- = \hbar \begin{pmatrix} 0 & 0 & 0 \\ \sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 \end{pmatrix}$$

$J_\pm = J_x \pm i J_y \Rightarrow$

$$\begin{cases} J_x = \frac{1}{2}(J_+ + J_-) = \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \\ J_y = \frac{1}{2i}(J_+ - J_-) = \frac{\hbar}{\sqrt{2}i} \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \end{cases}$$

$$J_x^2 = \frac{\hbar^2}{2} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}^2 = \frac{\hbar^2}{2} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

$$J_y^2 = -\frac{\hbar^2}{2} \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}^2 = -\frac{\hbar^2}{2} \begin{pmatrix} -1 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & -1 \end{pmatrix}$$

$$\Rightarrow J_x^2 - J_y^2 = \hbar^2 \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

For the doubly degenerate subspace of $\{|j=1, m=1\rangle, |j=1, m=-1\rangle\}$

$$\check{H} = \underbrace{A\hbar^2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}_{\hat{H}_0} + \underbrace{\Delta\hbar^2 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}_{\hat{H}'}$$

In the first order degenerate perturbation theory, $\check{H}'$ is diagonalized by $\frac{1}{\sqrt{2}}(|j=1, m=1\rangle \pm |j=1, m=-1\rangle)$ as basis with eigenvalue $\pm\Delta\hbar^2$

$$\Rightarrow E = \begin{cases} A\hbar^2 + \Delta\hbar^2 & : \frac{1}{\sqrt{2}}(|1,1\rangle + |1,1\rangle) \\ A\hbar^2 - \Delta\hbar^2 & : \frac{1}{\sqrt{2}}(|1,1\rangle - |1,-1\rangle). \end{cases}$$

//

Applied Quantum Mechanics Quals Problem

Robert Mawhinney
December 4, 2011

Tritium, ^{3}H, is highly radioactive and decays with a half-life of 12.3 years to ^{3}He by the emission of an electron and an electron anti-neutrino from its nucleus. The electron's average kinetic energy is 5.7 keV. Explain why its departure can be treated as sudden in the sense that the electron of the original tritium atom barely moves while the ejected electron leaves.

Calculate the probability that the newly-formed ^{3}He atom is in an excited state, by evaluating $\langle 1, 0, 0; Z = 2 | 1, 0, 0; Z = 1 \rangle$. Here the notation gives the n, l, m values for the electronic state.

Solutions

The binding energy of the H is just 13.6 eV and, by the virial theorem, its kinetic energy is half of this, so the speed of the ejected electron is larger by a factor of $\sqrt{5700/6.8} \approx 29 >> 1$. Hence the orbital electron barely moves in the time required for the ejected electron to get clear of the atom.

After the decay, the orbital electron is still in the ground state of H. The amplitude for it to be in the ground state of the new Hamiltonian is $\langle 1,0,0;Z=2|1,0,0;Z=1\rangle$. In the postiion representation, this is

$$\begin{aligned}
\langle 1,0,0;Z=2|1,0,0;Z=1\rangle &= \frac{1}{2}\left(\frac{4}{a_0}\frac{2}{a_0}\right)^{3/2}\int d^3x\, e^{-2r/a_0}\, Y_0^0\, e^{-r/a_0}\, Y_0^0 && (1)\\
&= 4\frac{2^{3/2}}{a_0^3}\int dr\, r^2\, e^{-3r/a_0} && (2)\\
&= \frac{4}{3^3}2^{3/2}\int dx\, x^2\, e^{-x} && (3)\\
&= \frac{4}{3^3}2^{3/2}2! && (4)\\
&= 0.838 && (5)
\end{aligned}$$

Alternatively, one can count the places where a_0 enters, since for ^{3}He, one will have $a_0' = a_0/2$. The inner product would be one for the ground state of H, but this is modified by

$$2^{3/2} * \frac{2^3}{3^3} = 0.838 \qquad (6)$$

The first factor is from the overall normalization of the wavefunction. The second comes from changing e^{-2r/a_0} for the H normalization integral by e^{-3r/a_0} for the overlap integral.

So the probability of being in an excited state is

$$P = 1 - |\langle 1,0,0;Z=2|1,0,0;Z=1\rangle|^2 = 1 - 64\times 8/27^2 = 0.298 \qquad (7)$$

General-Section 4: Applied Quantum Mechanics

Solution to Problem 3

(a) In the rest frame of the electron the proton is moving with velocity –**v,** that produces a magnetic field **B** ~ **L**. The coupling of this field to the intrinsic magnetic moment from **S** is the spin-orbit interaction H_{so}.

(b) Rewrite the spin-orbit interaction $H_{so} = A\mathbf{S}.\mathbf{L}$ as

$$H_{so} = (A/2)\,(J^2 - L^2 - S^2) \quad (1)$$

where J is the total angular momentum. The states with quantum numbers J, L, and S make this Hamiltonian diagonal. This states can be written as: $|j, m, l, s>$, where m are the components of j.

In this basis the matrix elements of H_{so} are:

$$<H_{so}> = <A(r)>_{nl}\,\hbar^2\,\{j(j+1) - l(l+1) - 3/4\} \quad (2)$$

(c) For n=2 the allowed values of l are 0 or 1. The allowed values of j are $l \mp 1/2$.

The value $j = 1/2$ is obtained from $l = 0,1$. There are two terms.

The value $j = 3/4$ is obtained from $l = 1$ only (one term here).

There are also two matrix elements:

$$<A(r)>_{2,0} = A_0 \quad (3)$$

and

$$<A(r)>_{2,1} = A_1 \quad (4)$$

The spin-orbit interaction energies are obtained with Eqs. (2) (4).

A hypothetical particle has negative squared mass such that we can write $m = i\mu$ where μ is a real quantity with units of mass. In all other respects, the particle satisfies the rules of special relativity, though sometimes in surprising ways. In answering the questions below you are encouraged to use natural units.

- Find expressions for the energy, E, and momentum magnitude, p of the particle in terms of its velocity β.

  There are multiple ways to evaluate these relations. The most robust is to use $\beta = p/E$ with $p \equiv |\vec{p}|$. Then, starting from $E^2 = p^2 + m^2 = p^2 - \mu^2$,

  $$E^2 = E^2\beta^2 - \mu^2 \Rightarrow E^2 = -\frac{\mu^2}{1-\beta^2}.$$

  Then, for E to be real, $1 - \beta^2 \leq 0$ or $\beta \geq 1$ (the limiting values at one will be addressed below). Then,

  $$E = \frac{\mu}{\sqrt{\beta^2 - 1}}. \tag{1}$$

  Similarly,

  $$\frac{p^2}{\beta^2} = p^2 - \mu^2 \Rightarrow p^2\left(\frac{1}{\beta^2} - 1\right) = -\mu^2.$$

  If $\beta > 1$, the factor multiplying p^2 is negative so we can absorb the minus sign on μ^2. Then

  $$p^2\left(1 - \frac{1}{\beta^2}\right) = \mu^2 \Rightarrow p^2\left(\beta^2 - 1\right) = \mu^2\beta^2$$

  so, finally

  $$p = \frac{\mu\beta}{\sqrt{\beta^2 - 1}} \tag{2}$$

- What are the maximum and minimum velocities for the particle and at what values of E and p are the maximum and minimum velocities attained?

  Part a shows that $\beta \geq 1$. From the results of part a, E clearly increases as $\beta \to 1$ from above, and at $\beta = 1$, $E = \infty$ and

$p = \infty$. As β increases, E and p both decrease monotonically with $E \to 0$ as $\beta \to \infty$. Since there is no natural cutoff on either E or β we must conclude that β can increase to ∞ at which the particle will have zero energy. However, the momentum approaches a constant value $p \to \mu$ as $\beta \to \infty$.

- Suppose a measurement of the velocity of the hypothetical particle yields $\beta - 1 \approx 2 \times 10^{-5}$ for $E \approx 30$ GeV. Estimate the value of μ that would be consistent with the measurement.

  We have $\beta = 1 + \delta$ with δ a small number compared to one. Then $1/\sqrt{\beta^2 - 1} \approx 1/\sqrt{2\delta}$. Using the value given,

  $$\frac{1}{\sqrt{2\delta}} \approx \frac{1}{2} \times 10^{5/2} \approx 0.5 \times 100 \times 3.$$

  So

  $$\mu \approx 2 \times (30 \text{ GeV})/300 = 0.2 \text{ GeV}. \tag{3}$$

- Consider the free propagation of our hypothetical particle in the "lab" frame at some velocity β. Suppose we define two events along the spacetime trajectory of the particle separated by a time Δt as observed in the lab frame. Show that it is possible to find another Lorentz frame in which the particle appears to be propagating backward in time. Evaluate the kinematics of the particle in that frame. How might an observer in that frame understand these results?

  Use the Lorentz transformation

  $$\Delta t' = \gamma_B \Delta t - \beta_B \gamma_B \Delta x,$$

  where β_B is the boost velocity and γ_B is the associated Lorentz factor, and where $\Delta x = \beta \Delta t$. Then,

  $$\Delta t' = \Delta t\ \gamma_B (1 - \beta_B \beta).$$

  Then, if

  $$\beta \beta_B > 1, \text{ or } \beta_B > 1/\beta \tag{4}$$

  (which can happen because $\beta > 1$) $\Delta t' < 0$ and it appears as though the order of events is reversed. But, in that same frame,

  $$E' = \gamma_B E - \beta_B \gamma_B p = E\ \gamma_B (1 - \beta \beta_B),$$

and if $\beta_B > 1/\beta$, $E' < 0$ as well! So to the observer in the new frame it looks like a negative energy particle propagating backward in time, equvalent (in a simplistic way, but also correctly) to a positive energy particle propagating forward in time. So, no violation of causality.

RELATIVITY

In this problem, please measure energies in MeV (mega-electron volts), velocities in units of the speed of light c and rest mass in units of MeV/c^2. Consider a collision between two particles, each of rest mass $m = 0.5 MeV/c^2$. In this problem I ask you to compare two reference frames: the center of mass reference frame and the 'lab' reference frame in which one of the two particles is not moving. Suppose that in the center of mass reference frame the two particles are initially moving along x with velocities $v = \pm 0.8c$. In the lab frame one particle has velocity $v = 0$ and the other particle has a velocity which is directed along negative x.

(a) Suppose that the collision is elastic, and that after the collision the two particles emerge moving along the $\pm y$ direction in the center of mass frame. Please give the angles made by the trajectories of the two particles with respect to the x axis, in the lab frame.

The transformation which takes the problem from the COM frame to the lab frame is a boost by velocity $v = 0.8c$ in the $+x$ direction. The relevant parts of the 4-momentum in the COM frame are

$$\begin{pmatrix} E \\ p_x \\ p_y \end{pmatrix} = \begin{pmatrix} mc^2\gamma_v \\ 0 \\ mv\gamma_v \end{pmatrix} \tag{1}$$

and the relevant parts in the lab frame are (denoting for the moment the velocity of the transformation by v_0

$$\begin{pmatrix} E \\ p_x \\ p_y \end{pmatrix} = \begin{pmatrix} mc^2\gamma_v\gamma_{v_0} \\ -mv_0\gamma_v\gamma_{v_0} \\ mv\gamma_v \end{pmatrix} \tag{2}$$

thus the angle is

$$\theta = \mp ArcTan\left[\frac{v\gamma_v}{v_0\gamma_v\gamma_{v_0}}\right] = ArcTan\frac{1}{\gamma_v} \tag{3}$$

where in the last equality I used $v_0 = v$

$$\theta = \mp ArcTan(0.36)$$

(b) Suppose now that what collides are a particle and its antiparticle, so that the two colliding particles annihilate and produce two photons which, in the center of mass frame, move off in the $\pm y$ direction. Please find the momenta of the two photons, in the lab frame.

Now the final momenta of the two photons are $\pm mc\gamma_v\hat{y}$ *but the kinematics of the Lorentz transformation is the same so we have*

$$\begin{aligned} p_y^{lab} &= p_y^{COM} = \pm mc\gamma_v \\ p_x^{lab} &= -\gamma_v^2 mv \end{aligned}$$

so

$$\theta = ArcTan\frac{mc\gamma_v}{mv\gamma_v^2} = ArcTan\frac{1}{.8*.6} \approx ArcTan(2)$$