Columbia University
Department of Physics
QUALIFYING EXAMINATION

Wednesday, January 11, 2012
1:00PM to 3:00PM
Modern Physics
Section 3. Quantum Mechanics

Two hours are permitted for the completion of this section of the examination. Choose 4 problems out of the 5 included in this section. (You will not earn extra credit by doing an additional problem). Apportion your time carefully.

Use separate answer booklet(s) for each question. Clearly mark on the answer booklet(s) which question you are answering (e.g., Section 3 (QM), Question 2, etc.).

Do **NOT** write your name on your answer booklets. Instead, clearly indicate your **Exam Letter Code**.

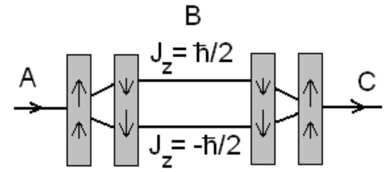
You may refer to the single handwritten note sheet on $8\frac{1}{2}$" $\times$ 11" paper (double-sided) you have prepared on Modern Physics. The note sheet cannot leave the exam room once the exam has begun. This note sheet must be handed in at the end of today's exam. Please include your Exam Letter Code on your note sheet. No other extraneous papers or books are permitted.

Simple calculators are permitted. However, the use of calculators for storing and/or recovering formulae or constants is NOT permitted.

Questions should be directed to the proctor.

Good Luck!

1. Consider the idealized experimental setup shown at the right. A beam of neutral, spin-1/2 atoms enters from the left (region A) moving with velocity v. This beam is separated into two parallel beams according to the atom's value of S_z by a region of inhomogeneous magnet field.



These two separated beams propagate to the right in region B and are then recombined by a second region of inhomogeneous magnetic field into a single beam which continues to move to the right in region C. The regions of magnetic field are arranged so that those atoms with $S_z = +\hbar/2$ are deflected into the upper path in region B and those with $S_z = +\hbar/2$ follow the lower path. The recombination occurs only if the value of S_z has not been changed in region B. Atoms whose J_z value has been changed in region B are absorbed before reaching region C. Assume that the incoming beam of atoms in region A is prepared so that each atom is in a spin eigenstate with $S_x = +\hbar/2$.

(a) If differences in the length of the paths followed by atoms with $S_z = \pm\hbar/2$ cause those in the upper path to acquire an additional phase ϕ relative to those in the lower path, find the average direction for the atomic spin of those atoms that are found in the right-moving beam in region C.

(b) If an additional region of uniform magnetic field $\vec{B} = B\hat{e}_z$ of length L is introduced into the upper path in region B, find the resulting average spin polarization for the atoms found in region C. (Assume that the atoms have gyromagnetic ratio γ and magnetic moment $\vec{\mu} = \gamma\vec{S}$.)

(c) If the direction of the magnetic field in part (b) is changed so that $\vec{B} = B\hat{e}_y$, find the factor by which the resulting intensity of the the beam in region C is reduced.

(d) A sufficiently intense beam of low frequency light (which does not affect the atoms' spins) illuminates only the lower path in region B, allowing an observer to identify each atom which passes along that path. What is the resulting polarization of the atoms in region C? (Assume that the magnetic field introduced in parts (b) and (c) is absent.)

2. A particle of mass m is contained within an impenetrable one-dimensional well extending from $x = -\frac{L}{2}$ to $x = +\frac{L}{2}$. The particle is in its ground state.

   (a) Find the eigenfunctions of the ground state and the first excited state.

   (b) The walls of the well are instantaneously moved outward to form a new well extending from $-L < x < +L$. Calculate the probability that the particle will stay in the ground state (of the new well configuration) during this sudden expansion.

   (c) Calculate the probability that the particle jumps from the initial ground state to the first excited final state.

3. Consider a Helium atom.

   (a) Compute the ground state energy of Helium assuming the electrostatic repulsion energy expectation $\Delta E(1s^2)$ of the two electrons can be treated in perturbation theory. (Hint: for fixed r_1, r_2 you can use $\int d\Omega_1(1/r_{12}) = 4\pi/r_>$ where $1/r_> = 1/r_1$ or $1/r_2$ depending on which is less. Also, $\int e^{-cx}\{1, x, x^2\}dx = -e^{-cx}\{1/c, (1+cx)/c^2, (2+2cx+c^2x^2)/c^3\}$).

   (b) What energy photons are emitted in the ${}^3P_1 \rightarrow {}^1S_0$ and ${}^1P_1 \rightarrow {}^1S_0$ transitions? Write down the integrals needed *without evaluation* of the first-order perturbation theory direct, $D(1s2p)$, and exchange energy, $E(1s2p)$, shifts that contribute to the electrostatic repulsion in the spin singlet and triplet cases. Assume that you are given that $D(1s2p) = 13$ eV and $E(1s2p) = 1$ eV and use part a.) for the $1s^2$ ground state.

4. Consider two identical electrons with spin-$\frac{1}{2}$ with mass m that are confined in 1-dimensional box whose length is L. The potential energy is 0 inside of the box and infinite outside of the box. The interaction between two particles can be expressed by the potential energy $V(x_1, x_2)$, where x_1 and x_2 are coordinates of two particles.

   (a) We first consider independent particles without interaction, *i.e.* $V(x_1, x_2) = 0$. What is the energy of the ground state? Write down the ground state wave function considering spatial and spin state symmetry.

   (b) Again, assuming non-interacting electrons ($V(x_1, x_2) = 0$), what is the energy and degeneracy of the first excited state? Write down explicit wave function considering spatial and spin state symmetry.

   (c) Now consider a simple electron-electron interaction of the form $V(x_1, x_2) = V_0\delta(x_1 - x_2)$. Assuming V_0 is small, find the correction for the ground state energy in (a) due to this *e*-*e* interaction. You do not have to evaluate integral explicitly in this problem. You may leave the closed form of integration.

   (d) Assuming the same simple interaction given in (c) (*i.e.* $V(x_1, x_2) = V_0\delta(x_1 - x_2)$), describe how degeneracy in the excited state discussed in (b) would be lifted.

5. For fully relativistic particles, the concept of potential energy is not truly applicable since there cannot be instantaneous action at a distance, as implied by a potential energy function. However, for velocities where the particle's motion is slightly relativistic (say, $v/c \sim 0.1$) the potential energy idea is approximately valid when the force responds fast enough to the particle's motion. Consider a harmonic oscillator satisfying these conditions and show that an approximate Hamiltonian for this slightly relativistic system is

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{m\omega^2}{2}\hat{x}^2 - \frac{1}{8c^2m^3}\hat{p}^4$$

Treating the $\hat{p}^4$ term as a perturbation, find

(a) $\langle n|\hat{p}^4|0\rangle$

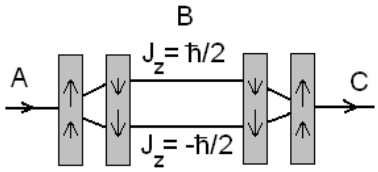
(b) The leading non-vanishing ground state energy shift

(c) The leading corrections to the ground state eigenvector $|0\rangle$

N. Christ December 12, 2011

Quals Problem

1. Consider the idealized experimental setup shown at the right. A beam of neutral, spin-1/2 atoms enters from the left (region A) moving with velocity v. This beam is separated into two parallel beams according to the atom's value of S_z by a region of inhomogenous magnet field.



These two separated beams propagate to the right in region B and are then recombined by a second region of inhomogenous magnetic field into a single beam which continues to move to the right in region C. The regions of magnetic field are arranged so that those atoms with $S_z = +\hbar/2$ are deflected into the upper path in region B and those with $S_z = +\hbar/2$ follow the lower path. The recombination occurs only if the value of S_z has not been changed in region B. Atoms whose J_z value has been changed in region B are absorbed before reaching region C. Assume that the incoming beam of atoms in region A is prepared so that each atom is in a spin eigenstate with $S_x = +\hbar/2$.

   (a) If differences in the length of the paths followed by atoms with $S_z = \pm\hbar/2$ cause those in the upper path to acquire an additional phase ϕ relative to those in the lower path, find the average direction for the atomic spin of those atoms that are found in the right-moving beam in region C.

   (b) If an additional region of uniform magnetic field $\vec{B} = B\hat{e}_z$ of length L is introduced into the upper path in region B, find the resulting average spin polarization for the atoms found in region C. (Assume that the atoms have gyromagnetic ratio γ and magnetic moment $\vec{\mu} = \gamma\vec{S}$.)

   (c) If the direction of the magnetic field in part (b) is changed so that $\vec{B} = B\hat{e}_y$, find the factor by which the resulting intensity of the the beam in region C is reduced.

   (d) A sufficient intense beam of low frequency light (which does not affect the atoms' spins) illuminates only the lower path in region B, allowing an observer to identify each atom which passes along that path. What is the resulting polarization of the atoms in region C? (Assume that the magnetic field introduced in parts (b) and (c) is absent.)

Suggested Solution

1. (a) When recombined each atom will have the spin wavefunction

$$|\psi\rangle = \frac{1}{\sqrt{2}}\left\{e^{i\phi}|+\frac{1}{2}\rangle + |-\frac{1}{2}\rangle\right\} \tag{1}$$

with the expectation values:

$$\begin{aligned} \langle S_x\rangle &= \frac{\hbar}{2}\cos(\phi) && (2)\\ \langle S_y\rangle &= -i\frac{\hbar}{2}\sin(\phi/2) && (3)\\ \langle S_z\rangle &= 0 && (4) \end{aligned}$$

(b) The answer will be the same as in part (a) except ϕ will be increased to $\phi+\theta$ where $\theta = B\gamma L/2v$.

(c) The spin will be rotated by the 2×2 matrix $\cos(\theta)+i\sigma_y\sin(\theta)$. This implies that the intensity of the transmitted beam will be reduced by those atoms in the upper beam that were lost because their spins were flipped, an overall factor of $1-\sin(\theta)^2/2$.

(d) The beam will be unpolarized.

Particle in Varying-Width Potential Well

A particle of mass *m* is contained within an impenetrable one-dimensional well extending from $x = -\frac{L}{2}$ to $x = +\frac{L}{2}$. The particle is in its ground state.

a) Find the eigenfunctions of the ground state and the first excited state.
b) The walls of the well are instantaneously moved outward to for a new well extending from $-L < x < +L$. Calculate the probability that the particle will stay in the ground state during this sudden expansion.
c) Calculate the probability that the particle jumps from the initial ground state to the first excited final state.

Solution:

a) The ground state for particle is:

$$\psi_0(x) = \sqrt{\tfrac{2}{L}}\cos\left(\tfrac{\pi x}{L}\right)$$

and first excited state:

$$\psi_1(x) = \sqrt{\tfrac{2}{L}}\sin\left(\tfrac{2\pi x}{L}\right)$$

b) After the expansion, the final eigenfunctions are, for the ground state:

$$\psi_0'(x) = \tfrac{1}{\sqrt{L}}\cos\left(\tfrac{\pi x}{2L}\right)$$

and for the first excited state:

$$\psi_1'(x) = \tfrac{1}{\sqrt{L}}\sin\left(\tfrac{\pi x}{L}\right)$$

In the sudden approximation, define P_{0j} as the probability that the particle starts in the ground state 0 and ends in the final state *j*, so that:

$$P_{0j} = \left|I_{0j}\right|^2$$

with:

$$I_{0j} = \int_{-L/2}^{L/2} dx\, \psi_j'(x)\psi_0(x)$$

So the amplitude for the particle to remain in the ground state is:

$$\begin{aligned} I_{00} &= \tfrac{\sqrt{2}}{L}\int_{-L/2}^{L/2} dx \cos\left(\tfrac{\pi x}{2L}\right)\cos\left(\tfrac{\pi x}{L}\right) \\ &= \tfrac{1}{L\sqrt{2}}\int_{-L/2}^{L/2} dx\left(\cos\left(\tfrac{\pi x}{2L}\right) + \cos\left(\tfrac{3\pi x}{2L}\right)\right) \\ &= \tfrac{1}{L\sqrt{2}}\tfrac{4L}{\pi}\left(\sin\left(\tfrac{\pi x}{2L}\right) + \tfrac{1}{3}\sin\left(\tfrac{3\pi x}{2L}\right)\right)\Big|_{x=L/2} \\ &= \tfrac{8}{3\pi} \end{aligned}$$

So the probability P_{00} is:

$$P_{00} = \left(\tfrac{8}{3\pi}\right)^2$$

c) For the transition between the initial ground state and the final excited state ψ_1', the amplitude for the transition is:

$$I_{01} = \int_{-L/2}^{L/2} dx\, \psi_1'(x)\psi_0(x)$$

$$= \tfrac{\sqrt{2}}{L} \int_{-L/2}^{L/2} dx \sin\left(\tfrac{\pi x}{L}\right)\cos\left(\tfrac{\pi x}{L}\right)$$

Since this is an odd function in *x* and we are integrating over $-\frac{L}{2} < x < +\frac{L}{2}$ then $I_{01} = 0$ and the probability P_{01} is:

$$P_{01} = 0$$

1. Quantum Mech section Quals 2013 :

   He atom

   a) Compute the ground state energy of Helium assuming the electrostatic repulsion energy expectation $\Delta E(1s^2)$ of the two electrons can be treated in perturbation theory. (Hint: for fixed r_1, r_2 you can use $\int d\Omega_1(1/r_{12}) = 4\pi/r_>$ where $1/r >= 1/r_1(1/r_2)$ depending on which is less. Also $\int e^{-cx}\{1, x, x^2\} = -e^{-cx}\{1/c, (1+cx)/c^2, (2+2cx+c^2x^2)/c^3\}$)

   b) What energy photons are emitted in the ${}^3P_1 \to {}^1S_0$ and ${}^1P_1 \to {}^1S_0$ transitions? Write down the integrals needed *without evaluation* of the first order perturbation theory direct, $D(1s2p)$, and exchange energy, $E(1s2p)$, shifts that contrinute to the electrostatic repulsion in the spin singlet and triplet cases. Assume that you are given that $D(1s2p) = 13$ eV and $E(1s2p) = 1$ eV and use part a for the $1s^2$ ground state.

Quantum: Two fermions in 1-dimensional box

Consider two identical fermions with spin ½ with mass *m* are confined in 1-diemsnional box whose length is *L*. The potential energy is 0 inside of the box and infinite outside of the box. The interaction between two particles can be expressed by the potential energy $V(x_1, x_2)$, where x_1 and x_2 are coordinates of two particles.

(a) We first consider independent particle without interaction, i.e. $V(x_1, x_2)=0$. What is the energy of the ground state? Write down the ground wave function considering spatial and spin state symmetry.

(b) Again without interaction ($V(x_1, x_2)=0$), what is the energy of the first excited state and number of degeneracy for non-interacting electrons? Write down explicit wave function considering spatial and spin state symmetry.

(c) We now consider the electron-electron interaction in a simple form of $V(x_1, x_2) = V_0\, \delta(x_1 - x_2)$. Assuming V_0 is small, find out the correction of energy in (a) due to this e-e interaction. You do not have to evaluate integral explicitly in this problem. You may leave the closed form of integration.

(d) With the consideration of $V(x_1, x_2)=0$ given in (c), describe how degeneracy in the excited state discussed in (b) would be lifted.

Two fermions in 1-D Box

For independent particles in 1-D Box ($0 \le x \le L$)

spatial wave function $\psi_n(x) = \sqrt{\frac{2}{L}} \sin(\frac{\pi}{L} n x)$

1-particle energy $E_n = \frac{\hbar^2}{2m}(\frac{\pi}{L})^2 n^2$

spin wave function $\sigma_\uparrow$, $\sigma_\downarrow$: spin ↑ or ↓.

(a) For the ground state, no interaction, considering antisymmetry

$$E_{GND} = E_1 + E_1 = 2E_1 = \frac{\hbar^2}{m}\left(\frac{\pi}{L}\right)^2$$

$$|\psi_{GND}\rangle = \psi_1(x_1)\psi_1(x_2)\frac{1}{\sqrt{2}}(\sigma_{1\uparrow}\sigma_{2\downarrow} - \sigma_{1\downarrow}\sigma_{2\uparrow})$$

(b) First excited state, no interaction

$$E_{1st} = E_1 + E_2 = \frac{5\hbar^2}{2m}\left(\frac{\pi}{L}\right)^2$$

- spin singlet state

$$|\psi_{1st}^{singlet}\rangle = \frac{1}{\sqrt{2}}\left[\psi_1(x_1)\psi_2(x_2) + \psi_2(x_1)\psi_1(x_2)\right]\frac{1}{\sqrt{2}}(\sigma_{1\uparrow}\sigma_{2\downarrow} - \sigma_{1\downarrow}\sigma_{2\uparrow})$$

- spin triplet state.

$$|\psi_{1st}^{triplet}\rangle = \frac{1}{\sqrt{2}}\left[\psi_1(x_1)\psi_2(x_2) - \psi_2(x_1)\psi_1(x_2)\right] \begin{cases} \sigma_{1\uparrow}\sigma_{2\uparrow} \\ \frac{1}{\sqrt{2}}(\sigma_{1\uparrow}\sigma_{2\downarrow} + \sigma_{1\downarrow}\sigma_{2\uparrow}) \\ \sigma_{1\downarrow}\sigma_{2\downarrow} \end{cases}$$

→ total degeneracy : $1 + 3 = \underline{4}$

(c)
$$\Delta E_{GND} = \langle \psi_{GND} | V_0 \delta(x_1 - x_2) | \psi_{GND} \rangle$$
$$= \left(\frac{2}{L}\right)^2 \int_0^L dx_1 \int_0^L dx_2 \sin^2\left(\frac{\pi}{L}x_1\right) \sin^2\left(\frac{\pi}{L}x_2\right) V_0 \delta(x_1 - x_2)$$
$$= \left(\frac{2}{L}\right)^2 V_0 \int_0^L \sin^4\left(\frac{\pi}{L}x\right) dx$$

(d)
$$\Delta E_{1st}^{(triplet)} = \langle \psi_{1st}^{(triplet)} | V_0 \delta(x_1 - x_2) | \psi_{1st}^{(triplet)} \rangle = 0$$
$$\Delta E_{1st}^{(singlet)} = \langle \psi_{1st}^{(singlet)} | V_0 \delta(x_1 - x_2) | \psi_{1st}^{(singlet)} \rangle = \int_0^L dx \cdot \frac{V_0}{2}\left(2\psi_1(x)\psi_2(x)\right)^2$$
$$= 2 \cdot \left(\frac{2}{L}\right)^2 V_0 \int_0^L \sin^2\left(\frac{\pi}{L}x\right) \sin^2\left(\frac{2\pi}{L}x\right) dx$$

Quantum Mechanics Quals Problem

Robert Mawhinney
December 1, 2011

For fully relativistic particles, the concept of potential energy is not truly applicable since there cannot be instantaneous action at a distance, as implied by a potential energy funtion. However, for velocities where the particle's motion is slightly relativistic (say, $v/c \sim 0.1$) the potential energy idea is approximately valid when the force responds fast enough to the particle's motion. Consider a harmonic oscillator satisfying these conditions and show that an approximate Hamiltonian for this slightly relativistic system is

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{m\omega^2}{2}\hat{x}^2 - \frac{1}{8c^2m^3}\hat{p}^4 \tag{1}$$

Treating the $\hat{p}^4$ term as a perturbation, find

1. $\langle n|\hat{p}^4|0\rangle$
2. The leading non-vanishing ground state energy shift
3. The leading corrections to the ground state eigenvector $|0\rangle$

Solutions

The kinetic term in the Hamiltonian comes from expanding $\sqrt{p^2c^2+m^2c^4}-mc^2$ as

$$mc^2\sqrt{1+\frac{p^2}{m^2c^2}}-mc^2=\frac{p^2}{2m}-\frac{1}{8m^3c^2}p^4+\cdots \tag{2}$$

which immediately yields the given Hamiltonian

Using

$$p=i\sqrt{\frac{m\hbar\omega}{2}}(a^\dagger-a) \tag{3}$$

gives

$$\langle n|\hat{p}^4|0\rangle = \left(\frac{m\hbar\omega}{2}\right)^2\langle n|(a^\dagger-a)^4|0\rangle \tag{4}$$

$$= \left(\frac{m\hbar\omega}{2}\right)^2\langle n|a^\dagger a^\dagger aa+\text{perms}-a^\dagger a^\dagger a^\dagger a+\text{perms}+(a^\dagger)^4|0\rangle \tag{5}$$

since terms with more powers of a vanish. The non-zero possibilities are

$$\langle 4|(a^\dagger)^4|0\rangle = \sqrt{4!}=2\sqrt{6} \tag{6}$$

$$\langle 2|a^\dagger a^\dagger aa^\dagger|0\rangle = \langle 2|a^\dagger a^\dagger|0\rangle=\sqrt{2} \tag{7}$$

$$\langle 2|a^\dagger aa^\dagger a^\dagger|0\rangle = \langle 2|a^\dagger a^\dagger aa^\dagger|0\rangle+\langle 2|a^\dagger a^\dagger|0\rangle=2\sqrt{2} \tag{8}$$

$$\langle 2|aa^\dagger a^\dagger a^\dagger|0\rangle = \langle 2|a^\dagger aa^\dagger a^\dagger|0\rangle+\sqrt{2}=3\sqrt{2} \tag{9}$$

$$\langle 0|aaa^\dagger a^\dagger|0\rangle = \langle 0|aa^\dagger aa^\dagger|0\rangle+1=2 \tag{10}$$

$$\langle 0|aa^\dagger aa^\dagger|0\rangle = 1 \tag{11}$$

This gives

$$\langle n|\hat{p}^4|0\rangle = \left(\frac{m\hbar\omega}{2}\right)^2\left(3\delta_{n,0}-6\sqrt{2}\delta_{n,2}+2\sqrt{6}\delta_{n,4}\right) \tag{12}$$

The leading, non-vanishing energy shift is

$$\delta E_0=-\frac{1}{8m^3c^2}\langle 0|\hat{p}^4|0\rangle=-\frac{3\hbar^2\omega^2}{64mc^2} \tag{13}$$

The correction to the ground state eigenvector is

$$\frac{V_{20}}{-2\hbar\omega}|2\rangle + \frac{V_{40}}{-4\hbar\omega}|4\rangle = -\frac{\hbar\omega}{64mc^2}\left(\frac{-6\sqrt{2}|2\rangle}{-2} + \frac{2\sqrt{6}|4\rangle}{-4}\right) \tag{14}$$

$$= \frac{\hbar\omega}{128mc^2}\left(-6\sqrt{2}|2\rangle + \sqrt{6}|4\rangle\right) \tag{15}$$