Columbia University
Department of Physics
QUALIFYING EXAMINATION

Monday, January 9, 2012
1:00PM to 3:00PM
Classical Physics
Section 1. Classical Mechanics

Two hours are permitted for the completion of this section of the examination. Choose 4 problems out of the 5 included in this section. (You will not earn extra credit by doing an additional problem). Apportion your time carefully.

Use separate answer booklet(s) for each question. Clearly mark on the answer booklet(s) which question you are answering (e.g., Section 1 (Classical Mechanics), Question 2, etc.).

Do **NOT** write your name on your answer booklets. Instead, clearly indicate your **Exam Letter Code**.

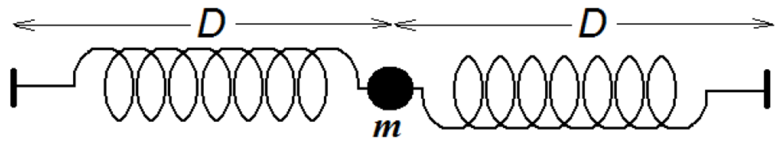
You may refer to the single handwritten note sheet on $8\frac{1}{2}$" $\times$ 11" paper (double-sided) you have prepared on Classical Physics. The note sheet cannot leave the exam room once the exam has begun. This note sheet must be handed in at the end of today's exam. Please include your Exam Letter Code on your note sheet. No other extraneous papers or books are permitted.

Simple calculators are permitted. However, the use of calculators for storing and/or recovering formulae or constants is NOT permitted.

Questions should be directed to the proctor.

Good Luck!

1. Two identical, massless springs of spring constant k and equilibrium length l are joined together and stretched between two fixed points separated by a distance $2D$ where $D > l$. A mass m is attached to the point where the springs are joined. Find the frequencies of small oscillation for the system, neglecting the effects of gravity. Note that the mass, m, is free to move in three dimensions.



2. Orbits around a black hole can be described in terms of the effective potential

$$V_{\text{eff}}(r) = -\frac{1}{r} + \frac{L^2}{2r^2} - \frac{L^2}{r^3}\,, \tag{1}$$

where L is the orbit's angular momentum. With respect to the classical Keplerian case, the only modification is the last $1/r^3$ term (for simplicity we are setting $G_N = 1$ and the reduced mass $\mu = 1$.) The above effective potential should be interpreted and used in the standard way, i.e., the radial equation of motion for a point particle orbiting a black hole is that associated with the Lagrangian

$$\mathcal{L} = \tfrac{1}{2}\dot{r}^2 - V_{\text{eff}}(r)\,. \tag{2}$$

(a) Show that for $L^2 < 12$ there are no circular orbits, whereas for $L^2 > 12$ there are *two*.

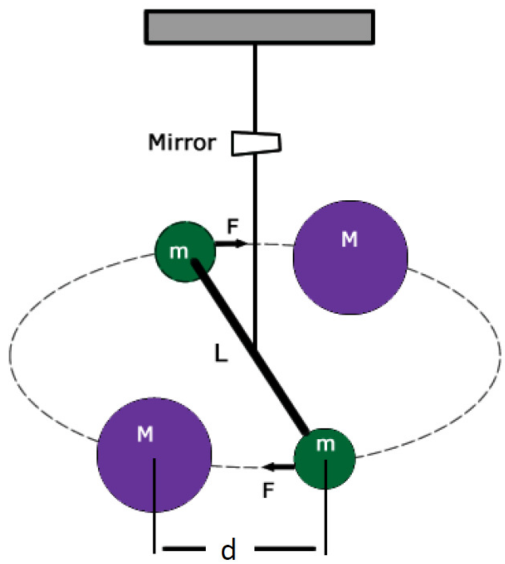
(b) Sketch a plot of $V_{\text{eff}}(r)$ for $L^2 < 12$ and for $L^2 > 12$.

(c) Describe qualitatively the possible orbits for $L^2 > 12$ and for $L^2 < 12$.

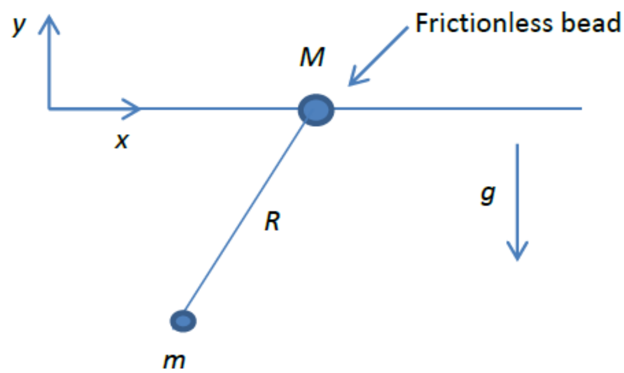
3. In the Cavendish experiment of 1797, a torsion balance was used to determine the first accurate value of the mass of the Earth or, alternately, the value of G, Newton's gravitational constant.

   Consider a torsion balance made of a rod of negligible mass and length L, suspended horizontally by its middle from a vertical wire. Small weights of equal mass m are attached at each end of the rod. In the first step of the experiment, it is observed that, when the rod is rotated through a small angle, thereby twisting the wire, and then released, the resulting torsion pendulum undergoes simple harmonic motion with a period T. Next, after the pendulum is stopped and is in its equilibrium position, a pair of large weights of equal mass M are placed on opposite sides of the rod, each one near one of the masses m. Due to the gravitational attraction only between each pair of masses (M, m), the rod is observed to rotate through an angle Θ and then come to rest in that position, with each mass M a distance d away from the corresponding mass m.

   Determine an expression for G in terms of the given variables of the problem.



4. A bead of mass M moves without friction along a stationary rod that lies along the x-axis. A pendulum of length R is suspended from this bead with a mass m at its end. The pendulum swings in the xy-plane, where y is the vertical axis. Assuming small oscillations, what is the frequency at which the pendulum oscillates?

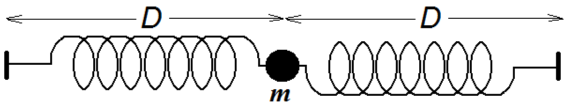


5. Find the net acceleration of a raindrop falling through a cloud. Assume that the raindrop is a tiny sphere, and the cloud consists of similar small droplets that are uniformly suspended at rest; when the falling drop hits a droplet within the cloud, the two merge into a combined, still spherical, drop, which continues falling.

N. Christ December 05, 2011

Quals Problem

1. Two identical, massless springs of spring constant k and equilibrium length l are joined together and stretched between two fixed points separated by a distance $2D$ where $D > l$. A mass m is attached to the point where the springs are joined. Find the frequencies of small oscillation for the system, neglecting the effects of gravity.



Suggested Solution

1. (a) Use the three coordinates x, y and z to locate the mass relative to an origin which is the equilibrium position of the mass. Choose z to be parallel to the equilibrium direction of the springs.

   (b) For small oscilation the motion in these three directions will be independent. The z motion is the easiest to determine with a restoring force which is double that produced by a single spring. The resulting Newton's law and frequency are given by

$$m\frac{d^2z}{dt^2} = -2kz \quad \text{which implies} \quad \omega = \sqrt{\frac{2k}{m}}$$

   (c) Since the springs are stretched they will produce a tension $T = k(D - l)$. Consider motion in the x direction. When m is displaced an amount x off the axis of symmetry, this tension will result in a restoring force of $2Tx/D$ giving the behavior

$$m\frac{d^2x}{dt^2} = -2k(D-l)\frac{x}{D} \quad \text{which implies} \quad \omega = \sqrt{\frac{2k(D-l)}{mD}}$$

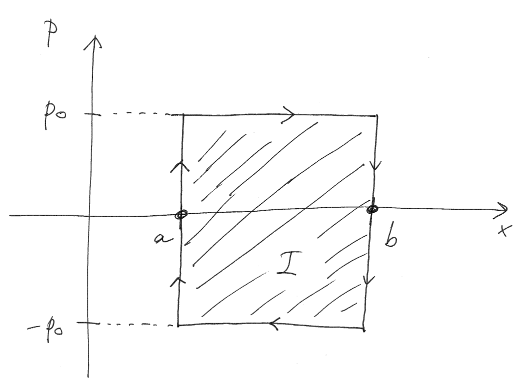
   Motion in the y direction will be identical.

1 General (Stat Mech): Single particle thermodynamics

Consider a point mass moving freely along x in the interval $[a, b]$. At the two extrema $x = a, b$ there are two walls. When our particle hits either wall, it bounces back elastically.

1. Draw the trajectory of the particle in *phase space*.
2. Suppose now that the two walls start moving, very slowly, as a result of some external forces. Compute the adiabatic invariant. [*Recall that the adiabatic invariant is the area enclosed by the orbit in phase space.*]
3. Given your answer to item 2, under what conditions is energy approximately conserved?
4. Explain in physical terms why your answer to item 3 makes sense. Is energy conserved in each collision?
5. If, as a result of their slow motion, the two walls eventually touch each other, what is the final energy of our particle?
6. Show that on time scales much longer than the particle 'round trip' time, this system obeys an ideal gas-like law,
$$P \cdot V = nk_BT \; . \tag{1}$$
What is the thermodynamic interpretation of the adiabatic invariant you computed in item 2?

Solution



1.
2. $I = \text{area} = 2(b - a)p_0$.
3. $E = H = p_0^2/(2m)$. Energy is approximately constant if p_0 is. Given the adiabatic invariant above, to have energy conservation we need $(b - a) = \text{const}$, i.e. the two walls have to move in the same direction at the same speed—not necessarily at constant speed though.

4. Energy is not conserved in each collision: if you bounce elastically off a receding wall, you end up with *less* energy than you started with. However, what you lose by this collision, you get back when you hit the other wall, if the two walls are moving a the same speed. (Recall that the conservation of the adiabatic invariant involves a time-average over a full cycle.) So, under the condition of item 2, the *average* energy is conserved.

5. Since $I = \text{const}$, when $(b-a) \to 0$ we have $p_0 \to \infty$, $E \to \infty$.

6. The natural definitions of pressure, temperature and volume for our system are,

$$P = \left\langle \left| \frac{dp}{dt} \right|_{\text{one wall}} \right\rangle = \frac{2p_0}{\tau} = \frac{p_0^2}{m(b-a)} \tag{2}$$

$$\tfrac{1}{2} k_B T = E = \frac{p_0^2}{2m} \tag{3}$$

$$V = (b-a) \ , \tag{4}$$

where τ denotes the round-trip time, $\langle \dots \rangle$ denotes time averaging over a few cycles, and of course the number of particles is one. We get precisely eq. (1).

Moving the walls adiabatically slowly, without injecting "heat" into the system, does not change entropy. Therefore, our adiabatic invariant should be associated with entropy in this thermodynamical interpretation of the system. We can work out the actual relation between the two quantities from the first law of thermodynamics,

$$dE = TdS - PdV \ , \tag{5}$$

which, according to the "dictionary" (2)–(4), gives

$$dS = k_B \Big(\frac{dp_0}{p_0} + \frac{dV}{V} \Big) \ . \tag{6}$$

On the other hand, just from $I = 2p_0 V$, we have

$$dI = 2\Big(V dp_0 + p_0 dV \Big) = I \frac{1}{k_B} dS \tag{7}$$

Integrating this ODE we get

$$S = k_B \log I + \text{const} \ . \tag{8}$$

2 Mechanics: Orbits around a black hole

Orbits around a black hole can be described in terms of the effective potential

$$V_{\text{eff}}(r) = -\frac{1}{r} + \frac{L^2}{2r^2} - \frac{L^2}{r^3} \ , \tag{9}$$

where L is the orbit's angular momentum. With respect to the classical Keplerian case, the only modification is the last $1/r^3$ term (for simplicity we are setting $G_N = 1$ and the reduced mass $\mu = 1$.) The above effective potential should be interpreted and used in the standard way, i.e., the radial equation of motion for a point particle orbiting a black hole is that associated with the Lagrangian

$$\mathcal{L} = \tfrac{1}{2} \dot{r}^2 - V_{\text{eff}}(r) \ . \tag{10}$$

1. Show that for $L^2 < 12$ there are no circular orbits, whereas for $L^2 > 12$ there are *two*.
2. Sketch a plot of $V_{\text{eff}}(r)$ for $L^2 < 12$ and for $L^2 > 12$.
3. Describe qualitatively the possible orbits for $L^2 > 12$ and for $L^2 < 12$.

Solution

1. Circular orbits are associated with the extrema of the effective potential. Their radius r_0 obeys

$$V'_{\text{eff}}(r_0) = \frac{1}{r_0^2} - \frac{L^2}{r_0^3} + 3\frac{L^2}{r_0^4} = 0 \tag{11}$$

Upon multiplying by r_0^4, this is a second order equation,

$$r_0^2 - L^2 r_0 + 3L^2 = 0 \ , \tag{12}$$

with discrimant

$$\Delta = L^2(L^2 - 12) \ . \tag{13}$$

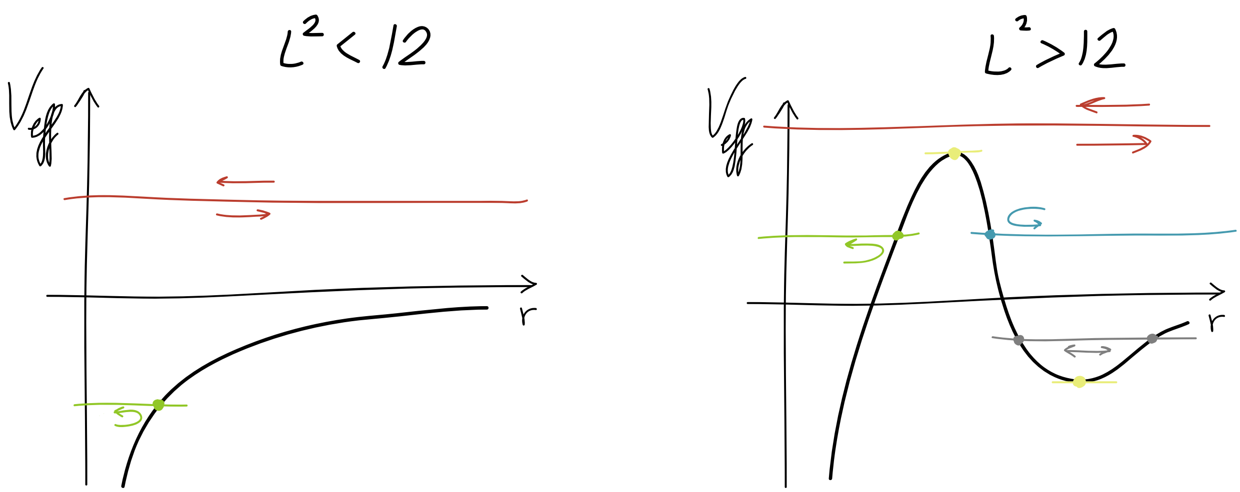
The two solutions are real (and distinct) if and only if $L^2 > 12$.

2. Given the information above about the extrema of V_{eff}, and the asymptotic behaviors

$$V_{\text{eff}}(r) \sim -\frac{1}{r} \qquad r \to \infty \tag{14}$$

$$V_{\text{eff}}(r) \sim -\frac{1}{r^3} \qquad r \to 0 \ , \tag{15}$$

we get the schematic plots shown in the figure.



3. As usual, in order to understand qualitatively the behavior of orbits one should draw constant-energy lines in the effective potential plot. Motion is allowed in regions where such lines are above the effective potential curve.

The $L^2 < 12$ case is very easy to analyze:

- $E > 0$: for an initially *outgoing* particle, the particle makes it to infinity an never comes back. For an incoming particle, the particle plunges into the black hole hitting $r = 0$.
- $E < 0$: there is a turning point, and all orbits eventually plunge into the black hole and hit $r = 0$.

The $L^2 > 12$ case has more possibilities. Starting from high energies:

- $E > \max V_{\text{eff}}$: same as the $E > 0$ case above.
- $E = \max V_{\text{eff}}$: circular orbit at the top of the peak. Unstable towards either going off to infinity, or to falling into the black hole, all the way to $r = 0$.
- $E < \max V_{\text{eff}}$, left of the peak: same as the $E < 0$ case above.
- $0 < E < \max V_{\text{eff}}$, right of the peak: there is a turning point. Incoming particles come in from infinity, reach a minimum radius, and then go back to infinity (and never come back).
- $\min V_{\text{eff}} < E < 0$, right of the peak: there are two turning points. Orbits are bound, although not closed (periodic), in general.
- $E = \min V_{\text{eff}}$, right of the peak: circular orbit at the bottom of the depression. Stable.

Mechanics Question

In the Cavendish experiment of 1797, a torsion balance was used to determine the first accurate value of the mass of the Earth or, alternately, the value of G, Newton's gravitational constant.

Consider a torsion balance made of a rod of negligible mass and length L, suspended horizontally by its middle from a vertical wire. Small weights of equal mass m are attached at each end of the rod. In the first step of the experiment, it is observed that, when the rod is rotated through a small angle, thereby twisting the wire, and then released, the resulting torsion pendulum undergoes simple harmonic motion with a period T. Next, after the pendulum is stopped and is in its equilibrium position, a pair of large weights of equal mass M are placed on opposite sides of the rod, each one near one of the masses m. Due to the gravitational attraction between each pair of masses (M, m), the rod is observed to rotate through an angle θ and then come to rest in that position, with each mass M a distance d away from the corresponding mass m.

Determine an expression for G in terms of the given variables of the problem.

Answer

Period of torsion pendulum is (1) $T = 2\pi\sqrt{(I/\kappa)}$ where κ is torsion constant of wire, and the torsion pendulum's rotational inertia is (2) $I = m(L/2)^2 + m(L/2)^2 = mL^2/2$.

Torque from wire is (3) $\tau = \kappa\theta$, which must be balanced by torque from gravity.

Torque from gravity is (4) $\tau = F_{grav} \times L/2 \times 2 = F_{grav} L = (GMm/d^2) L$

Equating (2) and (3) and solving for G one finds (5) $G = L\kappa\theta d^2/Mm$.

Solving for κ from equations (1) and (2) and plugging it into (5), one gets the final answer $G = 2\pi^2 L\theta d^2/MT^2$

E Weinberg

Mechanics #4 — Solution

① $M \to$ at $(x, 0)$

$m \to$ at $(x + R\sin\theta, \ -R\cos\theta)$

$KE = \frac{M}{2}\dot{x}^2 + \frac{m}{2}\{(\dot{x} + R\dot{\theta}\cos\theta)^2 + (R\dot{\theta}\sin\theta)^2\}$

$= \frac{1}{2}(M+m)\dot{x}^2 + \frac{m}{2}R^2\dot{\theta}^2 + m\dot{x}\dot{\theta}R\cos\theta$

$PE = -mgR\cos\theta$

$\mathcal{L} = T - V = \frac{1}{2}(M+m)\dot{x}^2 + \frac{m}{2}R^2\dot{\theta}^2 + m\dot{x}\dot{\theta}R\cos\theta + mgR\cos\theta$

Small θ approx

$\Rightarrow \mathcal{L} = \frac{1}{2}(M+m)\dot{x}^2 + \frac{m}{2}R^2\dot{\theta}^2 + mR\dot{x}\dot{\theta} - \frac{MgR}{2}\theta^2 + \text{const}$

x eq $\Rightarrow (M+m)\ddot{x} + mR\ddot{\theta} = 0$

θ eq $\Rightarrow mR^2\ddot{\theta} + mR\ddot{x} = -mgR\theta$

$\Rightarrow \left(mR^2 - \frac{m^2R^2}{M+m}\right)\ddot{\theta} = \frac{mMR^2}{M+m}\ddot{\theta} = -mgR\theta$

$\Rightarrow$ ang freq $\omega = \sqrt{\frac{g}{R}}\sqrt{\frac{M+m}{M}}$

CLASSICAL PHYSICS – MECHANICS

Raindrop falling through a cloud. SOLUTION.

The fall of the raindrop is a series of *completely inelastic* collisions. Therefore, conservation of mechanical energy does not apply; conservation of linear momentum, however, applies. Moreover, in this problem the mass of the projectile is increasing in time.

To get started, we assign the following variables and constants: $v(t)$, $r(t)$, and $m(t)$ are the velocity, radius, and mass of the raindrop as functions of time, and ρ and σ are the average mass densities of the raindrop and of the cloud. We can write three dynamics equations,

$$dm/dt = (3m/r)dr/dt, \tag{1}$$

$$dm/dt = \pi r^2 v\sigma, \tag{2}$$

$$mg = v(dm/dt) + m(dv/dt), \tag{3}$$

where eq. (1) arises from the mass-radius relationship for a shpere, eq. (2) is a statement of the mass increase as the raindrop sweeps out a volume through the cloud, and eq. (3) is equivalent to $F = dp/dt$. The problem requires finding dv/dt.

Note that $dm/dt = 4\pi r^2 \rho dr/dt$, hence (along with eq. (2)) we obtain both $v = 4(\rho/\sigma)dr/dt$ and

$$dv/dt = 4(\rho/\sigma)d^2r/dt^2. \tag{4}$$

Combining eqs. (3, 4, 1), we find

$$(\sigma/\rho)gr = 12(dr/dt)^2 + 4r(d^2r/dt^2). \tag{5}$$

If C is a numerical constant, the solution to eq. (5) must take the form

$$r(t) = C(\sigma/\rho)gt^2 \tag{6}$$

(as seen, for example, from dimensional analysis, and the fact that eq. (5) only depends on the parameter $(\sigma/\rho)g$). We determine $C = 1/56$ by direct substitution of eq. (6) into eq. (5). Differentiating twice, we find

$$d^2r(t)/dt^2 = (\sigma/\rho)g/28. \tag{7}$$

From eq. (4) it then follows that

$$dv/dt = g/7. \tag{8}$$

Note that the answer (8) is independent of all assumed parameters.