Columbia University
Department of Physics
QUALIFYING EXAMINATION

Friday, January 14, 2011
3:10PM to 5:10PM
General Physics (Part II)
Section 6.

Two hours are permitted for the completion of this section of the examination. Choose 4 problems out of the 6 included in this section. (You will not earn extra credit by doing an additional problem). Apportion your time carefully.

Use separate answer booklet(s) for each question. Clearly mark on the answer booklet(s) which question you are answering (e.g., Section 6 (General Physics), Question 2, etc.).

Do **NOT** write your name on your answer booklets. Instead, clearly indicate your **Exam Letter Code**.

You may refer to the single handwritten note sheet on $8\frac{1}{2}$" × 11" paper (double-sided) you have prepared on General Physics. The note sheet cannot leave the exam room once the exam has begun. This note sheet must be handed in at the end of today's exam. Please include your Exam Letter Code on your note sheet. No other extraneous papers or books are permitted.

Simple calculators are permitted. However, the use of calculators for storing and/or recovering formulae or constants is NOT permitted.

Questions should be directed to the proctor.

Good Luck!

1. A neutron star is composed primarily of neutrons packed to nuclear densities, and the mass of a neutron star is 1 to 3 times the mass of the sun.

   (a) Roughly calculate the radius of a neutron star.

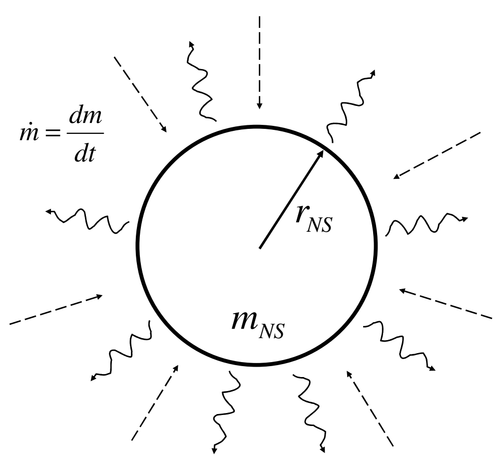
   A neutron star will often accrete matter onto its hard surface from a companion star. Assume this accretion flow is spherically symmetric and is falling onto the neutron star surface at a rate of approximately $dm/dt = 10^{-9}$ solar masses per year.

   (b) Estimate the total accretion luminosity emitted by a neutron star at the given accretion rate.

   (c) Assuming the neutron star radiates like a black body, what is the typical energy of the emitted photons? What type of telescope would be required to observe these photons?

   The accreting matter is subject not only to the force of gravity, but to a radiation pressure force from the emitted photons. Assume spherical symmetry and that the accreting matter is pure 100% ionized hydrogen.

   (d) Show that the above considerations lead to a critical luminosity – the Eddington luminosity – such that if the neutron star luminosity is higher than this value then the accretion flow will turn off. Find an algebraic and numerical estimate of this luminosity.



2. Deuterium's energy levels are shifted relative to those of hydrogen due to the nuclear mass difference. Find this 'isotope shift'. Are the deuterium transition wavelengths shorter or longer than the corresponding hydrogen wavelengths?

3. Consider a polyethylene chain:

`-CH=CH-CH=CH- . . .`

and represent its stretching vibrations by a linear chain of identical masses M connected by alternating springs of force constants K_1 and K_2. K_1 is for the `CH=CH` pair and K_2 is for the `CH-CH` pair. The separation of identical `-CH` units and of identical `=CH` units are both a.

(a) Write the equations of motion for stretching modes.

(b) Obtain the frequencies of normal modes as function of wave vector [$\omega = \omega(k)$].

(c) Sketch the dispersions $\omega(k)$ by obtaining the values of ω for the wave vectors $k \to 0$ and $k \to \pi/a$.

(d) In a Debye model the dispersion of the lowest (acoustical) branch ω is approximated by a linear dispersion $\omega(k) = c_s k$, where c_s is the speed of sound in the chain:

i. Evaluate c_s.

ii. Obtain the expression for the heat capacity in the limit of $T \to 0$.

4. Consider the nuclear shell model and "magic" numbers $Z^* = 2, 8, 20, 28, 50$ corresponding to filled shells.

   (a) An analytic nuclear potential model is a 3D isotropic harmonic oscillator with a given $\hbar\omega_0$. Draw and label the energy level diagram including the lowest 4 levels. Label each level with sets of integers, $(n, \ell_1, \ell_2, \ldots)$, where $n = n_1 + n_2 + n_3 \geq 0$, and $\ell_i \geq 0$, are allowed orbital angular momentum quantum numbers. Indicate by square brackets the total degeneracy, $[g(n)]$ of the level including any spin degrees of freedom. For example, the second level $(n = 1, \ell = 1)$ $[g = 6]$ corresponds to the $2P$ state in spectroscopic notation. Derive a general formula for $g(n)$ and use parity invariance to select the allowed ℓ_i for each level. Determine the magic numbers, $Z^*(3DHO)$, for this 3D HO nuclear potential model and compare to the nuclear data and to atomic electron Coulomb magic numbers.

   (b) Show that introducing a nuclear spin-orbit interaction of the form $V_{so} = \hbar\omega_1 \vec{S} \cdot \vec{L}$ splits the levels and reduces the degeneracy of the new levels level and lowers the $n = 3$ $1F_{j=7/2}$ levels in a way that could account for the unusual magic number, 28, of nuclei. First compute the splitting of the $n + 1$ level in the $|n, \ell, s, j, j_z>$ basis. Make a schematic qualitative diagram of the split energy levels labeled by (n, ℓ, j) and their degeneracy $[g(n, j)]$.

5. We have an ideal gas of atoms at chemical potential $\mu = -1$ eV and a temperature given by $kT = 0.1$ eV. The gas is in equilibrium with a metal surface with isolated binding sites for the atoms. Each binding site can adsorb (attach) 0, 1, or 2 atoms:

   - The energy with 0 adsorbed atoms is $E = 0$.
   - The energy with 1 adsorbed atom is $E = -1$ eV.
   - The energy with 2 adsorbed atoms is $E = -1.9$ eV.

   (a) Determine the probability that a site has no adsorbed atoms.

   (b) Determine the average number of adsorbed molecules at each site.

   (c) If we keep the temperature of the gas the same, should we increase or decrease its pressure to have equal probabilities of 1 and 2 atoms being adsorbed at a given site? Explain your answer qualitatively.

   (d) Now find the ratio by which we must increase or decrease the pressure of the gas (held at constant temperature) to have equal probabilities of 1 and 2 atoms being adsorbed at a given site.

6. Suppose you are doing a spectroscopy experiment on a cloud of ^{84}Kr atoms at room temperature and negligible pressure. You want to tune a laser to probe a transition at 811.51 nm (with respect to an atom at rest) by passing the laser through the gas in one direction. To make your measurements, you would like to use an old interferometer you found hidden in the bottom of a drawer in your lab. It has a resolving power of 10^7.

   Will this interferometer be good enough? Estimate the dominant relative uncertainty you would expect for this measurement. Suppose you could first cool the cloud to 1 K. Estimate the expected improvement in your results, if any.

General: Astrophysics, Hailey Answer

a.) The neutron star density is very roughly $m_n / V_n = \rho_{NS}$ and since $r_n \simeq 1.2$ fm, $m_n \simeq 1.7 \times 10^{-24}$ g

$$\rho_{NS} = m_n / \tfrac{4}{3}\pi r_n^3 \simeq 2 \times 10^{14}\ \text{g/cm}^3$$

Then $M_{NS} = \rho_{NS} \cdot V_{NS} = \frac{4}{3}\pi R_{NS}^3 \rho_{NS}$

$M_{NS} \simeq 2 \times M_{sun} \simeq 4 \times 10^{33}$ g $\Rightarrow$ Ans

$R_{NS} \simeq 1.7 \times 10^6$ cm $\simeq 17$ Km. This is close to the current range of $\sim 9-12$ Km.

b.) The gravitational potential energy (assuming $r \to \infty$ and at rest) released on hitting the surface is just GM/R per unit mass or

$$\frac{dE}{dt} = L = \frac{GM\dot{m}}{R} = 6.7 \times 10^{-8} \times 2 \times 2 \times 10^{33}\ \text{g} \times 10^{-9} \times \frac{2 \times 10^{33}\ \text{g}}{\pi \times 10^7\ \text{s}} \Big/ 1.7 \times 10^6\ \text{cm}$$

$L \simeq 10^{37}$ erg/s Ans

c) emitting uniformly as a black-body

$$L = 4\pi R_{NS}^2 \sigma T_{NS}^4 \qquad \sigma = 5.7\times10^{-5} \text{ cgs}$$

$$T_{NS} = \left(\frac{L}{4\pi R_{NS}^2 \sigma}\right)^{1/4} = \left\{\frac{10^{37}\text{ erg/s}}{4\pi \cdot 5.7\times10^{-5} \cdot (1.7\times10^6\text{ cm})^2}\right\}^{1/4}$$

$$T_{NS} \simeq 8\times10^6 \text{ K} \simeq 10^7 \text{ K}$$

or $KT_{NS} \simeq 1 \text{ keV}$.

The peak of the black-body is $\simeq 3KT$ or

$E_{\text{typical}} \simeq 3\text{ keV}$,

you need an X-ray telescope.

d.) The force of the photons on impacting the electrons is $p = E/c \quad \dot{p} = F = \frac{\dot{E}}{c} = \frac{L}{c}$

$$\text{Force/Area} = \text{radiation pressure} = \frac{L}{4\pi r^2 c}$$

$$\text{Force/electron} = \frac{L\sigma_T}{4\pi r^2 c}$$ (we ignore proton radiation pressure because $\sigma_T(m_p) << \sigma_T(m_e)$)

$$\text{Grav. Force} = \frac{GMm_p}{r^2}$$ (we ignore electron)

The electric force couples the e-p system.

Note both $\sim 1/r^2$ so there is a crucial Luminosity

$$\frac{L_e \sigma_T}{4\pi c} = GMm_p$$

02 $L_{Eddington} = \frac{G M m_p 4\pi c}{\sigma_T} \approx 2\times10^{38}\ \frac{erg}{s}$

using $\sigma_T \sim \frac{8}{3}\pi r_0^2 \quad r_0 \simeq 2.8\times10^{-13}\ cm$

$= 6.6\times10^{-25}\ cm^2$

$L_{edd} \approx 2\times10^{38}\ erg/s$ Ans

GENERAL PHYSICS – ATOMIC

Hydrogen vs. deuterium: isotope shift. SOLUTION.

Let μ be the reduced mass of the bound electron, and m and M the electron and nuclear masses, respectively. The hydrogenic eigenenergy with the nth principal quantum number is

$$E_n = -\frac{1}{2}\mu c^2 \alpha^2 \frac{1}{n^2} = -\frac{1}{2}\frac{\alpha^2}{n^2} mc^2 \frac{M}{M+m} \approx -\frac{1}{2}\frac{\alpha^2}{n^2} mc^2 \left(1 - \frac{m}{M}\right). \tag{1}$$

Thus, the ratio of the deuterium and hydrogen eigenenergies is

$$\frac{E_n^D}{E_n^H} \approx \frac{1 - m/M_D}{1 - m/M_H} \approx 1 + \frac{m_e}{2M_H} \simeq 1 + \frac{1}{3700}. \tag{2}$$

Since the lower n states of D are *more negative* relative to those of H by larger amounts (in absolute energy units) than the higher n states, the atomic transition frequencies of D are higher, hence the *wavelengths are shorter*.

General: condensed matter

Solution

(a) the equations of motion for stretching modes are

$M\ddot{u}_n = K_1 u_{n+1} - (K_1 + K_2) u_n + K_2 u_{n-1}$

$M\ddot{u}_{n-1} = K_2 u_n - (K_1 + K_2) u_{n-1} + K_1 u_{n-2}$

(b) the frequencies of normal modes as function of wave vector are

$\omega^2 = [(K_1+K_2)/M] \{1 \pm [1 - 4K_1K_2\sin^2(0.5\ ka)/(K_1 + K_2)^2]^{1/2}\}$

(c) the dispersions $\omega(k)]$ are sketched by obtaining the values of ω for the wave vectors $k\rightarrow0$ and $k\rightarrow\pi/a$ from the result in (b).

(d) In a Debye model the dispersion of the lowest (acoustical) branch ω is approximated by a linear dispersion $\omega(k)=c_sk$, where c_s is the speed of sound in the chain:

(i) $c_s = [K_1K_2 / M(K_1+K_2)] (a/2^{1/2})$

(ii) The expression for the heat capacity at constant volume V in the limit of $T\rightarrow0$ is

$C_V = 12\pi^4 Nk_B(T/\theta)^3$

where:

N is the number of units, k_B is the Boltzmann constant and

$\theta = (\hbar c_s/k_B).(6\pi^2 N/V)$

Solution

(a) The energy difference is the Fermi energy: $\mathcal{E}_F = \frac{\hbar^2 k_F^2}{2m}$ where $k_F^2 = (2\pi n)$

For $n_{low} = 10^{10}\ cm^{-2}$ $\mathcal{E}_F = 3.8 \times 10^{-17}$ ergs
$= 2.37 \times 10^{-2}$ meV

(b) Here there are two j-states populated. The Fermi energies of the two states have to be identical;

$$\frac{1}{2}E_2 + \frac{\hbar^2(2\pi n_0)}{2m} = \frac{3}{2}E_2 + \frac{\hbar^2}{2m}(2\pi n_1)$$

$$n_0 - n_1 = mE_2 / \hbar^2\pi = 4.22 \times 10^{11}\ cm^{-2}$$

$$n_0 + n_1 = n_{high} = 10^{12}\ cm^{-2}$$

$$n_0 = 0.71 \times 10^{12}\ cm^{-2};\ n_1 = 0.29 \times 10^{12}\ cm^{-2}$$

$\mathcal{E}_F = 1.68$ meV from the j = 0 level

(c) $k_B T = 8.6 \times 10^{-1}$ meV $>> \mathcal{E}_F$

The electron gas has a classical distribution energies.

1. General section Quals 2011: The nuclear shell model and "magic" numbers $Z^* = 2, 8, 20, 28, 50$ corresponding to filled shells.

   a) [20] An analytic nuclear potential model is a 3D isotropic harmonic oscillator with a given $\hbar\omega_0$. Draw and label the energy level diagram including the lowest 4 levels. Label each level with sets of integers, $(n, \ell_1, \ell_2, ..)$, where $n = n_1 + n_2 + n_3 \geq 0$, and $\ell_i \geq 0$, are allowed orbital angular momentum quantum numbers. Indicate by square brackets the total degeneracy, $[g(n)]$ of the level including any spin degrees of freedom. For example, the second level $(n = 1, \ell = 1)$ $[g = 6]$ corresponds to the $2P$ state in spectroscopic notation. Derive a general formula for $g(n)$ and use parity invariance to select the allowed ℓ_i for each level. Determine the magic numbers, $Z^*(3DHO)$, for this 3D HO nuclear potential model and compare to the nuclear data and to atomic electron Coulomb magic numbers.

   b) [20] Show that introducing a nuclear spin-orbit interaction of the form $V_{so} = -\hbar\omega_1 \vec{S} \cdot \vec{L}$ splits the levels and reduces the degeneracy of the new levels level and lowers the $n = 3$ $1F_{j=7/2}$ levels in a way that could account for the unusual magic number, 28, of nuclei. First compute the splitting of the $n+1$ level in the $|n, \ell, s, j, j_z >$ basis. Make a schematic qualitative diagram of the split energy levels labeled by (n, ℓ, j) and their degeneracy $[g(n, j)]$.

Solution part a:

The 3D HO Schroedinger equation seperates into three 1D HO problems and thus the energy levels can be labeled by 3 integers (n_1, n_2, n_3) with $0 \leq n_i$. Rotation invariance means that $L^2 = \ell(\ell+1)$ is conserved and thus we can also label states with (n, ℓ), where $n = n_1+n_2+n_3 \geq 0$ and $\ell \geq 0$. The distinct energy levels $E(n) = \hbar\omega_0(n + \frac{3}{2})$. The ground state has $n = 0$ in this notation. The degeneracy $g(n)$ follows from fact that given n, each $0 \leq n_i \leq n$. There are $n + 1$ allowed choices for n_1, leaving $n - n_1 + 1$ for n_2 and n_3 completely fixed once n, n_1, n_2 are fixed. Taking into acount the factor 2 degeneracy due to spin 1/2, $g(n) = 2\sum_{n_1=0}^{n}(n - n_1 + 1) = (n + 1)(n + 2)$. In the (n, ℓ_i) labeling on the other hand, there is a $2(2\ell_i + 1)$ degeneracy per ℓ_i. Parity conservation provides a restriction on the allowed ℓ_i since the angular eigenfunctions $Y_{\ell,m}(\theta, \phi)$ have parity $P = (-1)^{\ell}$. Since each 1D HO wavefunction has parity $(-1)^{n_i}$ the parity of level n is $(-1)^n$. Thus for n even, only even ℓ are allowed, while for n odd ,only odd ℓ allowed. Since $g(n) = \sum_{parity\ allowed\ \ell} 2(2\ell + 1)$, simple numerical evaluation gives $g(n) = 2, 8, 12, 20, 30$ for the $1S(n = 0, \ell = 0)[g = 2]$, $2P(1, 1)[6]$,$(2S, 1D)(2, \ell = 0, 2)[12]$, and $(2P, 1F)(3, \ell = 1, 3)[g = 20]$. The first 4 shells can accomodate $2 + 6 + 12 + 20 = 40$ protons. For the 3D HO, $Z* = 2, 8, 10, 40, 70$ in disagreement with nuclear data beyond the third shell. In atomic physics the ideal $V = -Z\alpha/r$ Coulomb potential leads to filled shells of Nobel gases $N_e = 2, 2 + (2 + 6) = 10, 2 + 8 + (2 + 6 + 10) = 38$. Multi-electron interactions change 38 to 36.

Solution part b:

With spin-orbit only L^2, S^2, J^2 and J_z are good quantum numbers for each n and in the $|n, \ell, s, j, j_z >$ basis,

$$< \ell, j|V_{so}|\ell, j > /\hbar\omega_1 = -\tfrac{1}{2}(j(j+1) - \ell(\ell+1) - 3/4) = \begin{cases} -\ell/2 & j = \ell + \frac{1}{2} \\ +(\ell+1)/2 & j = \ell - \frac{1}{2} \end{cases}$$

Therefore the higher $j = \ell + \frac{1}{2}$ with degereracy $g(\ell, j = \ell + \frac{1}{2}) = 2\ell + 2$ is shifted down (for positive ω_1 by $-\hbar\omega_1\ell$ while the lower $j = \ell - \frac{1}{2}$ with degeneracy $g(\ell, j = \ell - \frac{1}{2}) = 2\ell$ is shifted upwards by $+\hbar\omega_1(\ell+1)$. The energy of the $1F_{7/2}$ levels with degeneracy $g(3, 7/2) = 8$ is lowered to

$$E(n = 3, \ell = 3, j = 7/2) = \hbar\omega_0(3 + 3/2) - \frac{3}{2}\hbar\omega_1$$

while the $1F_{5/2}$ is raised to

$$E(n = 3, \ell = 3, j = 9/2) = \hbar\omega_0(3 + 3/2) + 2\hbar\omega_1$$

In contrast the $(3, 1, 3/2)2P_{3/2}$ levels are only lowered to

$$E(n = 3, \ell = 1, j = 3/2) = \hbar\omega_0(3 + 3/2) - \tfrac{1}{2}\hbar\omega_1$$

while the $2P_{1/2}$ is raised to

$$E(n = 3, \ell = 1, j = 1/2) = \hbar\omega_0(3 + 3/2) + \hbar\omega_1$$

Thus there is a gap $\hbar\omega_1$ between the $2P_{3/2}$ and the $1F_{7/2}$ with $g(7/2) = 8$ that could account for the fact that the observed fourth closed shell leads to a magic number 28 rather than 40 as in the 3D HO model. (In practice the nuclear potential is more like a square well than a HO and this also contributes to the splitting of ideal $2P$ and $1F$ degeneracy to enhance the gap.)

Solution to statistical mechanics problem

(a) The probability for any state s for a system with thermal and diffusive equilibrium is determined by the Gibbs factor: $P(s) = \exp\{-[E(s) - \mu N(s)]/kT\}/\mathcal{Z}$, with grand partition function $\mathcal{Z} = \Sigma_s \exp\{-[E(s) - \mu N(s)]/kT\}$. Consider the system to consist of one adsorption site in thermal and diffusive equilibrium with the gas. We then have three distinct states $s = 0, 1, 2$, as indicated above, and

P(no atoms)= $P(s = 0) = 1/\mathcal{Z} \exp[-(0+1\text{eV}\times 0)/kT] = 1/\mathcal{Z} \exp(0) = 1/\mathcal{Z}$, where

$$\mathcal{Z}=\exp[-(0+1\text{eV}\times0)/kT]+\exp[-(-1\text{eV}+1\text{eV}\times1)/kT]+\exp[-(-1.9\text{eV}+1\text{eV}\times2)/kT]= 2+e^{-1}$$

and P(no atoms) $= 1/(2+e^{-1}) = e/(2e+1) = 0.422$

(b) $\overline{N} = \Sigma_s N(s)P(s) = 0\times1/\mathcal{Z} + 1\times1/\mathcal{Z} + 2\times e^{-1}/\mathcal{Z} = (1+2e^{-1})/(2+e^{-1}) = (e+2)/(2e+1) = 0.578$

(c) From the above, we have

$P(1 \text{ atom}) = P(s=1) = 1/\mathcal{Z} \quad > \quad P(2 \text{ atoms}) = P(s=2) = e^{-1}/\mathcal{Z}$

To shift the balance towards absorption of two atoms at the surface site, we need to increase the density of atoms in the gas phase, i.e., to increase the gas pressure.

(d) To make the probabilities of adsorbing one or two atoms equal, we need to change the chemical potential μ so that $P(s=1) = P(s=2)$:

$\exp[-(-1\text{eV} - \mu)/kT] = \exp[-(-1.9\text{eV} - 2\mu)/kT]$ or $\mu = -0.9$ eV.

This implies that we need to change the chemical potential by $\Delta\mu = 0.1$ eV. For an ideal gas, the chemical potential varies with pressure at constant temperature as $\Delta\mu = kT \ln P_f/P_i$. Then for $\Delta\mu = kT = 0.1$ eV, we need to increase the pressure by a factor of e.

[Derivation of $\Delta\mu = kT \ln P_f/P_i$: From the thermodynamic identify $dG = -SdT + VdP + \mu dN$. For a process at constant T and N, we obtain $dG/dP = V = NkT/P$, where the later equality holds for an ideal gas. Since $G = N\mu$, we have $d\mu/dP = kT/P$, whence the desired relation.]

Spectroscopy Solution: The spectral peaks will be significantly broadened due to Doppler broadening, i.e. the thermal velocity of the gas cloud. The effect on the resolution can be approximated by considering the fractional Doppler shift.

First calculate the average speed at room temperature and at 1 K. At or below room temperature, the thermal energy is safely approximated as non-relativistic, so

$$\begin{aligned} 3/2k_BT &= 1/2mv^2 \\ \Rightarrow v &= \sqrt{3k_BT/m} \end{aligned}$$

Note: The above is actually the RMS velocity. For a Maxwell-Boltzmann distribution, the most probable speed is $\sqrt{2k_BT/m}$, but this is only small correction, and can be neglected in this approximation.

Recall that $k_B \approx 1.4 * 10^{-23}$ m^2 kg s^{-2} K^{-1}, and at room temperature, T$\approx$ 300K

$$\begin{aligned} \Rightarrow v_{T=300} &\approx 300m/s \\ \Rightarrow v_{T=1} &\approx 20m/s \end{aligned}$$

Recall the Doppler shift formula

$$\omega' = \gamma\omega(1 - \beta Cos(\theta))$$

is well approximated at non-relativistic velocities as

$$\omega' = \omega(1 - \beta Cos(\theta)) = \omega - \vec{k} \cdot \vec{v}$$

It's magnitude is maximum for $\theta \in \{0, \pi\}$. So, for $\omega' = \omega_0 = 2\pi c/\lambda_0$, we find that the magnitude of the fractional Doppler broadening is approximated to first order by

$$\frac{\omega - \omega_0}{\omega_0} = \frac{v}{c}$$

To improve the approximation, the absorption probability would need to be taken into account, but this changes the result by less than a factor of 2.

We thus find that the fractional Doppler shift is $\approx 10^{-6}$ at room temperature, and $\approx 6.7 * 10^{-8}$ at 1 K. So, resolving powers of roughly 10^6 and $1.5 * 10^7$ would be necessary to differentiate between Doppler broadened peaks.

Hence, at room temperature, the relative uncertainty of our measurement will be dominated by the width of the Doppler peak to $\approx 10^{-6}$. In this case, our old interferometer is no problem, as it can resolve such wide peaks. But, at 1 K, the relative uncertainty due to the resolution of the interferometer will be on the same order as that from the Doppler broadening, at $\approx 10^{-7}$, and thus can not be neglected. So, a better interferometer may help at the lower temp, but not much. Regardless, a factor of 10 improvement is expected when decreasing the temperature from 300 K to 1 K.