Columbia University
Department of Physics
QUALIFYING EXAMINATION

Wednesday, January 12, 2011
3:10PM to 5:10PM
Modern Physics
Section 4. Relativity and Applied Quantum Mechanics

Two hours are permitted for the completion of this section of the examination. Choose 4 problems out of the 5 included in this section. (You will not earn extra credit by doing an additional problem). Apportion your time carefully.

Use separate answer booklet(s) for each question. Clearly mark on the answer booklet(s) which question you are answering (e.g., Section 4 (Relativity and Applied QM), Question 2, etc.).

Do **NOT** write your name on your answer booklets. Instead, clearly indicate your **Exam Letter Code**.

You may refer to the single handwritten note sheet on $8\frac{1}{2}$" $\times$ 11" paper (double-sided) you have prepared on Modern Physics. The note sheet cannot leave the exam room once the exam has begun. This note sheet must be handed in at the end of today's exam. Please include your Exam Letter Code on your note sheet. No other extraneous papers or books are permitted.

Simple calculators are permitted. However, the use of calculators for storing and/or recovering formulae or constants is NOT permitted.

Questions should be directed to the proctor.

Good Luck!

1. Consider an electron gas confined in three dimensions whose energy can be related to its momentum by:

$$E\left(k_x, k_y\right) = \frac{\hbar^2}{2}\left(\frac{k_x^2}{m_x} + \frac{k_y^2}{m_y} + \frac{k_z^2}{m_z}\right)$$

where m_x, m_y, and m_z are the effective carrier mass along the x, y, and z direction, respectively.

   (a) Find the density of states $N(E_F)$ of this electron gas as a function of Fermi energy E_F.

   We now apply a magnetic field H. In Pauli paramagnetism, the electron energy changes linearly: $E \pm g\mu_B H$, where the $+/-$ sign depends on the spin direction, μ_B is the Bohr magneton, and g is the electron g-factor.

   (b) Let $n_\uparrow$ and $n_\downarrow$ be the spin up and down electron densities, respectively. Using the result obtained in (a), write down an expression for the magnetization $M = g\mu_B(n_\uparrow - n_\downarrow)$ at a finite temperature T. [You do not need to evaluate the integral for this part of the problem.]

   (c) Let us now consider the low-temperature limit, $k_B T \ll E_F$ in (b). Show that the M obtained in part (b) is linearly proportional to H for low magnetic field ($g\mu_B H \ll E_F$). What is the proportionality coefficient?

2. Consider a hydrogen atom in its ground state. The hyperfine interaction between the magnetic moment of the proton and the magnetic moment of the electron is written as:

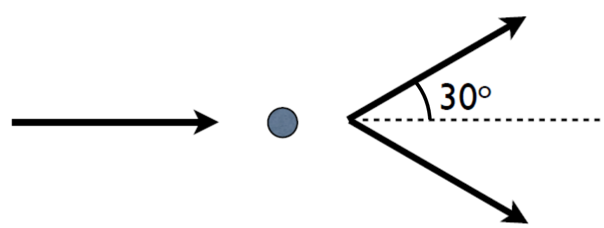
$$H_{hf} = A\vec{S}_1 \cdot \vec{S}_2 \tag{1}$$

In Equation 1, $\vec{S}_1$ is the spin of the electron and $\vec{S}_2$ is the spin of the proton.

The ground state of the hydrogen atom is split by the hyperfine coupling. Obtain the energies of these split levels.

3. Show that a constantly accelerating object would asymptotically approach the speed of light, in accordance with relativity. By constant acceleration we mean the following: whereas the velocity obeys $-(dt/d\tau)^2 + (dx/d\tau)^2 = -1$ (τ is the proper time and one spatial dimension is assumed), the acceleration obeys $-(d^2t/d\tau^2)^2 + (d^2x/d\tau^2)^2 = \alpha^2$, where α is a constant.

4. An electron with momentum p initially traveling along x collides with an electron at rest. After the collision the final velocity of one of the electrons is oriented at an angle of 30° with respect to the x axis and both electrons have the same energy. Find the momentum of this outgoing electron, in terms of the rest energy of the electron.



5. A steady laser beam with energy density U shines on a perfect mirror placed perpendicular to the beam. The mirror has mass m and area A all of which is illuminated by the beam. The mirror experiences no forces except for the radiation pressure. It is initially at rest (at time $t = 0$).

   (a) Find the force f applied to the mirror at $t = 0$.

   (b) Over time, the radiation pressure accelerates the mirror to a relativistic velocity v. What is the force applied by the beam to the mirror when the mirror is moving with $\beta \equiv v/c = 3/4$?

   (c) How long will it take to accelerate the mirror to a given Lorentz factor $\gamma \gg 1$?

Applied Quantum

(a) Considering volume of an ellipsoide in (K_x, K_y, K_z)

where ~~the~~ major axis is given by

$$K^0_{x,y,z} = \sqrt{\frac{2E_{Fi}}{\hbar^2} m_{x,y,z}}$$

Electron density

$$n = 2 \cdot \left(\frac{1}{2\pi}\right)^3 \frac{4\pi}{3} K^0_x K^0_y K^0_z$$

$$= \frac{1}{6\pi^2} \left[m_x m_y m_z\right]^{\frac{1}{2}} \left(\frac{2E_{Fi}}{\hbar^2}\right)^{3/2}$$

Density of state

$$N(E_{Fi}) = \left.\frac{dn}{dE}\right|_{E=E_{Fi}} = \frac{1}{4\pi^2}\left[m_x m_y m_z\right]\left[\frac{2}{\hbar}\right]^{3/2} E_F^{1/2}$$

(b) D.O.S of spin ↑ & ↓,

$$N_\uparrow(E) = \frac{1}{2} N(E + g\mu_B H)$$

$$N_\downarrow(E) = \frac{1}{2} N(E - g\mu_B H)$$

$$n_{\uparrow,\downarrow} = \int_{-\infty}^{\infty} dE \; N_{\uparrow,\downarrow}(E) f(E) \quad \text{where} \quad f(E) = \frac{1}{e^{\beta(E-\mu)}+1}$$

$$M = g\mu_B (n_\uparrow - n_\downarrow) = \mu_B \int_{-\infty}^{\infty} dE \left[N_\uparrow(E) - N_\downarrow(E)\right] f(E)$$

$$= \frac{\mu_B}{2} g \int_{-\infty}^{\infty} dE \left[N(E + g\mu_B H) - N(E - g\mu_B H)\right] f(E)$$

(c) For $g\mu_B H \ll E_F$

$$N(E+g\mu_B H) - N(E-g\mu_B H) \simeq 2 g\mu_B H \left(\frac{\partial N}{\partial E}\right)$$

$$\Rightarrow \quad M \simeq \frac{g\mu_B}{2} \int_{-\infty}^{\infty} dE \cdot 2 g\mu_B H \left(\frac{\partial N}{\partial E}\right) f(E)$$

$$= (g\mu_B)^2 H \int_{-\infty}^{\infty} \left(\frac{\partial N}{\partial E}\right) f(E)$$

$$= (g\mu_B)^2 H \int_{-\infty}^{\infty} N(E) \left(-\frac{\partial f}{\partial E}\right) \quad \left(-\frac{\partial f}{\partial E}\right) \simeq \delta(E-E_F)$$

integral by part ↑

$$= (g\mu_B)^2 N(E_F) \cdot H$$

General-Section 4: applied quantum mechanics

Solution

Total angular momentum is:

$$\mathbf{S} = \mathbf{S_1} + \mathbf{S_2}$$

For $S_1 = ½$ and $S_2 = ½$ the two states are:

singlet with S=0; and

triplet with S=1

Write:

$$S^2 = (\mathbf{S_1} + \mathbf{S_2})^2 = S_1^2 + S_2^2 + 2\mathbf{S_2}.\mathbf{S_2}$$

$$H_{hf} = A\mathbf{S_1}.\mathbf{S_2} = (A/2)(S^2 - S_1^2 - S_2^2) \qquad (1)$$

In Equation (1) $\mathbf{S_1}$ is the spin of the electron and $\mathbf{S_2}$ is the spin of the proton.

The needed eigenvalues are:

For S^2	$\hbar^2 S(S+1)$	(2)
For S_1^2	$\hbar^2 S_1(S_1+1)$	(3)
For S_2^2	$\hbar^2 S_2(S_2+1)$	(4)

From Equations (1) to (4) it follows that the hyperfine energies are:

for the singlet $-3\hbar^2 A/4$

for the triplet $\hbar^2 A/4$

Sec 4 - # 3

Answer: the simplest solution would be to observe that $t = \alpha^{-1}\sinh[\alpha\tau]$ and $x = \alpha^{-1}\cosh[\alpha\tau]$ satisfies both of the above equations, and argue that $dx/dt = \tanh[\alpha\tau]$ approaches unity in the large τ limit.

A more elaborate approach would be to use the first equation to write $d\tau = dt\sqrt{1-v^2}$, and rewrite the second equation as

$$\frac{d}{d\tau}\left[\frac{dt}{d\tau} - \frac{dx}{d\tau}\right]\frac{d}{d\tau}\left[\frac{dt}{d\tau} + \frac{dx}{d\tau}\right] = -\alpha^2 \tag{1}$$

which can be rewritten as

$$\frac{(1+g^2)^2}{4g^2}\frac{1}{g^2}\left(\frac{dg}{dt}\right)^2 = \alpha^2 \tag{2}$$

where $g \equiv \sqrt{(1-v)/(1+v)}$. Taking the square root of this (with the appropriate sign) and integrate, one obtains $2\alpha t = g^{-1} - g$ plus integration constant. Hence, as $t \to \infty$, $v \to 1$.

Solution: (units c=1) After collision, the two outgoing particles have same energy and must have equal and opposite values of components of momentum perpendicular to x, thus they must have the same value of momentum along x and this must be equal to the initial momentum divided by 2. Call this momentum p_x.

1/3 credit for getting to this point

Conservation of energy:

$$\sqrt{4p_x^2 + m^2} + m = 2\sqrt{p_x^2 + p_y^2 + m^2} \tag{1}$$

Squaring, cancelling and squaring again

$$m^2 p_x^2 = p_y^2 m^2 + p_y^4 \tag{2}$$

Thus knowledge of relation between p_x and p_y fixes angle and determines magnitude of momentum.

1/3 credit for getting to this point

Momentum $\vec{p} = m\gamma\vec{v}$ so

$$p_y = m\gamma_v v sin\frac{\pi}{6} = |p| sin\frac{\pi}{6} = tan\frac{\pi}{6} p_x \tag{3}$$

Thus

$$m^2 p^2 cos^2\frac{\pi}{6} = m^2 p^2 sin^2\frac{\pi}{6} + p^4 sin^4\frac{\pi}{6} \tag{4}$$

or

$$p^2 = m^2 \frac{\left(cos^2\frac{\pi}{6} - sin^2\frac{\pi}{6}\right)}{sin^4\frac{\pi}{6}} = 8m^2 \tag{5}$$

Solution:

(a) The density of momentum in the beam is U/c. The momentum impacting the static mirror during time dt is $dP_{\rm in} = (cdt\,A)\,(U/c)$. The corresponding momentum carried by the reflected photons is $dP_{\rm out} = -dP_{\rm in}$, so the mirror gains momentum $dP = 2dt\,A\,U$. The applied force is $f = dP/dt = 2AU$.

(b) The momentum impacting the moving mirror during time dt is

$$dP_{\rm in} = (c - v)dt\,A\,\frac{U}{c}.$$

Consider the reflection of one photon with initial momentum $p_{\rm in}$. The corresponding four-momentum is $(p_{\rm in}, p_{\rm in}, 0, 0)$. Lorentz transformation of this four-vector yields the momentum measured in the rest-frame of the mirror: $p'_{\rm in} = p_{\rm in}\gamma(1 - \beta)$. In this frame, the photon after reflection has the momentum $p'_{\rm out} = -p'_{\rm in}$. Transforming $p'_{\rm out}$ back to the lab frame, one finds how the reflection has changed the photon momentum: $p_{\rm out} = \gamma(p'_{\rm out} + \beta p'_{\rm out}) = -\gamma^2(1 - \beta)^2 p_{\rm in}$.

The reflection of many photons with net momentum $dP_{\rm in}$ gives

$$dP_{\rm out} = -\gamma^2(1-\beta)^2\,dP_{\rm in} = -\frac{1-\beta}{1+\beta}\,dP_{\rm in}.$$

The force applied to the mirror is

$$f = \frac{dP_{\rm in} - dP_{\rm out}}{dt} = \left(1 + \frac{1-\beta}{1+\beta}\right)\frac{dP_{\rm in}}{dt} = 2\left(\frac{1-\beta}{1+\beta}\right)AU = \frac{2}{7}\,AU.$$

[A faster way is available if one is familiar with the transformation of stress-energy tensor, which gives $U' = \gamma^2(1-\beta)^2 U$. Then use $f' = 2AU'$ and $f = f'$.]

(c) The mirror accelerates according to the dynamic equation $dP_m/dt = f$, where $P_m = m\gamma v$ is the momentum of the mirror. This gives

$$mc\frac{d}{dt}(\gamma\beta) = 2\left(\frac{1-\beta}{1+\beta}\right)AU.$$

When $\gamma \gg 1$, this equation simplifies to

$$mc\frac{d\gamma}{dt} = \frac{AU}{2\gamma^2} \qquad \Rightarrow \qquad t = \frac{2mc\gamma^3}{3AU}.$$