Columbia University
Department of Physics
QUALIFYING EXAMINATION
Friday, January 15, 2010
3:10 PM - 5:10 PM

General Physics (Part II)
Section 6.

Two hours are permitted for the completion of this section of the examination. Choose **4 problems** out of the 6 included in this section. Remember to hand in **only** the 4 problems of your choice (if by mistake you hand in 5 or 6 problems, the highest scoring problem grade(s) will be dropped). Apportion your time carefully.

Use separate answer booklet(s) for each question. Clearly mark on the answer booklet(s) which question you are answering (e.g., Section 6 (General Physics), Question 2; Section 6 (General Physics), Question 6; etc.)

Do **NOT** write your name on your answer booklets. Instead clearly indicate your **Exam Letter Code**.

You may refer to the single handwritten note sheet on 8 $1/2 \times 11$" paper (double-sided) you have prepared on General Physics. The note sheet cannot leave the exam room once the exam has begun. This note sheet must be handed in at the end of today's exam. Please include your Exam Letter Code on your note sheet. No other extraneous papers or books are premitted.

Simple calculators are permitted. However, the use of calculators for storing and/or recovering formulae or constants is NOT permitted.

Questions should be directed to the proctor.

Good luck!!

1. Consider a self-gravitating, infinitely extended fluid (i.e. a fluid whose individual volume elements interact gravitationally with each other). Assume that the fluid has negligible pressure. The relevant equations are the continuity and Euler ones for the fluid's dynamics, and the Poisson equation for the Newtonian potential. Call ρ the fluid's density, $\vec{v}$ its velocity field, and Φ the gravitational potential per unit mass.

   (a) Show that there is a solution to the dynamics such that ρ is homogeneous and the fluid expands radially, with $\vec{v}$ proportional to the position vector:

   $$\vec{v}(\vec{x}, t) = H(t)\vec{x} \ .$$

   (b) Determine $\rho(t)$ and $H(t)$, via a power-law ansatz $\rho \propto t^\alpha$, $H \propto t^\beta$.

   (c) Show that, despite the appearances, for this solution the origin is not a preferred point. That is, all observers comoving with the fluid see exactly the same fluid flow around them. Are there other possible solutions with the same property?

2. Rydberg atoms are highly excited atoms, usually with the principal quantum number $n \gg 1$.

   (a) Find the energy spacing between the nth and the $(n+1)$st Rydberg states of hydrogen.

   (b) Find the size of the atom in the nth energy state.

   (c) Are relativistic effects more or less important in Rydberg states than in the low-lying states? (In other words, how do the typical electron velocities compare to the speed of light in both cases?)

3. A scientist constructed a field effect transistor (FET) that employs a two-dimensional electron gas. In the FET the density of electrons is varied with an external voltage. Under the working conditions of this device, the energy states of the electrons can be represented as:

$$\mathbf{E}(j, k_x, k_y) = \left[j + \frac{1}{2}\right] \mathbf{E}_z + \frac{\hbar^2}{2m}(k_x^2 + k_y^2)$$

The two-dimensional electron gas is in the x-y plane. $\left[j + \frac{1}{2}\right] \mathbf{E}_z$ represents the energy for the motion of electrons along the direction normal to the plane. $\mathbf{E}_z = 0.001$ eV (1 eV $= 1.6 \times 10^{-19}$ Joules). The allowed values of the quantum number j are $j = 0, 1, 2, \ldots$. k_x and k_y are the two in-plane components of the wave vector of states for the electrons, and m is the electron rest mass ($m = 9.11 \times 10^{-28}$ gm).

When the areal electron density of the two-dimensional electron gas is controlled by an external gate voltage, the density can have two limiting values: n_{low} and n_{high}.

(a) Assume that $n_{\text{low}} = 10^{10}$ cm^{-2}. Find the difference between the energies of the lowest and highest states that are populated by the electrons when the temperature is T = 0.

(b) Repeat (a) with $n_{\text{high}} = 10^{12}$ cm^{-2}.

(c) Repeat (a) for T = 10 K.

(The Boltzmann constant is $k_B = 10^{-23}$ Joules/K.)

4. (a) How many photons per second are emitted by a typical incandescent light bulb?

   (b) How many photons per second reach your eye, if you are standing 1 km away from the light bulb?

   (c) Can you see the light bulb from 1 km away, if about 10% of the photons are in the visible portion of the spectrum, 10% of the photons reaching the eye actually hit the retina, and the minimum flux to activate the brain response is 100 photons/s?

5. A fluid in thermodynamic equilibrium at temperature T fills a rigid cubical container of volume V. For wavelenths relevant to the questions below assume the sound speed in the fluid (v_s) to be independent of wavelenght λ ($\ll$ interatomic spacing).

   (a) What is the lowest angular frequency (ω_0) for a standing sound wave in the fluid?

   (b) What is the average energy in that mode when $k_B T \gg \hbar\omega_0$?
   (k_B is the Boltzmann constant, $\hbar \equiv h/2\pi$, neglect "zero point" energy)

   (c) What is the average energy in that mode when the inequality in (b) does not hold?

   (d) What is the probability for finding no energy in this mode (neglecting "zero point" energy)?

   (e) What is the total energy in the modes whose wavelength lies between λ and $\lambda + d\lambda$ and are $\ll V^{1/3}$?

6. The relationship between the free energy F, the internal energy U, temperature T, and entropy S of a gas with a fixed number of atoms is given by:

$$F = U - TS$$

(a) Find an expression for pressure P and entropy S expressed as partial derivatives with respect to the free energy.

(b) Write an expression for $\left(\frac{\partial S}{\partial V}\right)_T$ in terms of pressure, volume, and temperature.

(c) Use the result from part (b) to show that

$$\left(\frac{\partial U}{\partial T}\right)_V = -T\left(\frac{\partial P}{\partial T}\right)_V \left(\frac{\partial V}{\partial T}\right)_S$$

(d) Show that

$$\left(\frac{\partial T}{\partial V}\right)_S = \frac{\left(\frac{\partial V}{\partial T}\right)_P \left(\frac{\partial P}{\partial V}\right)_T}{\left(\frac{\partial S}{\partial T}\right)_V}$$

Astro - Nicolis.

2 General: Astrophysics – the Hubble flow

Consider a self-gravitating, infinitely extended fluid. Assume that the fluid has negligible pressure. The relevant equations are the continuity and Euler ones for the fluid's dynamics, and the Poisson equation for the Newtonian potential. Call ρ the fluid's density, $\vec{v}$ its velocity field, and Φ the gravitational potential per unit mass.

Sec 6
General II
1
astrophysics

Atomic Zelevinsky
Sec. 6
general II
2 (atomic)

GENERAL PHYSICS – ATOMIC

Rydberg hydrogen atoms.

Rydberg atoms are highly excited atoms, usually with the principal quantum number $n \gg 1$.
a) Find the energy spacing between the nth and the $(n+1)$st Rydberg states of hydrogen.
b) Find the size of the atom in the nth energy state.
c) Are relativistic effects more or less important in Rydberg states than in the low-lying states? (In other words, how do the typical electron velocities compare to the speed of light in both cases?)

Finczuk
Sec. 6 General II
3 condensed matter

General-Section 6: condensed matter

A scientist constructed a field effect transistor (FET) that employs a two-dimensional electron gas. In the FET the density of electrons is varied with an external voltage. Under the working conditions of this device the energy states of the electrons can be represented as:

$$\mathbf{E}(j, k_x, k_y) = \mathbf{E_z}\,[j + \tfrac{1}{2}] + (\hbar^2/2m)\,(k_x^2 + k_y^2)$$

The two-dimensional electron gas is in the (x,y) plane.
$\mathbf{E_z}$ (j+ ½), represents the energy for motion of electrons along the direction normal to the plane. $\mathbf{E_z}$ = 0.001eV (1eV=1.6x10^{-19}Joules). The allowed values of the quantum number j are j= 0, 1, 2 .... . k_x and k_y are the two in-plane components of wave vector of the states of the electrons, and m is the electron rest mass (m=9.11x10^{-28}gm).

When the areal electron density of the two-dimensional electron gas is controlled by an external gate voltage, the density can have two limiting values: n_{low} and n_{high}.

(a) Assume that n_{low} = 10^{10}cm^{-2}. Find the difference between the energies of the lowest- and highest- states that are populated by the electrons when the temperature is T=0.

(b) Repeat (a) with n_{high} =10^{12}cm^{-2}.

(c) Repeat (a) for T=10K (the Boltzmann constant is k_B=10^{-23}Joules/K).

Solution

(a) The energy difference is the Fermi energy: $E_F = \frac{\hbar^2 k_F^2}{2m}$ where $k_F^2 = (2\pi n)$

For $n_{low} = 10^{10}\ cm^{-2}$ $E_F = 3.8 \times 10^{-17}$ ergs
$= 2.37 \times 10^{-2}$ meV

(b) Here there are two j-states populated. The Fermi energies of the two states have to be identical;

$$\frac{1}{2}E_2 + \frac{\hbar^2(2\pi n_0)}{2m} = \frac{3}{2}E_2 + \frac{\hbar^2}{2m}(2\pi n_1)$$

$$n_0 - n_1 = m E_2 / \hbar^2 \pi = 4.22 \times 10^{11}\ cm^{-2}$$

$$n_0 + n_1 = n_{high} = 10^{12}\ cm^{-2}$$

$$n_0 = 0.71 \times 10^{12}\ cm^{-2};\ n_1 = 0.29 \times 10^{12}\ cm^{-2}$$

$E_F = 1.68$ meV from the $j=0$ level

(c) $k_B T = 8.6 \times 10^{-1}$ meV $>> E_F$

The electron gas has a classical distribution energies.

order of Magnitude Zelevinsky
Sec. 6 General II
#4
order of mag.

GENERAL PHYSICS – ORDER OF MAGNITUDE ESTIMATE

Light bulbs and photons.

a) How many photons per second are emitted by a typical incandescent light bulb?

b) How many photons per second reach your eye, if you are standing 1 km away from the light bulb?

c) Can you see the light bulb from 1 km away, if about 10% of the photons are in the visible portion of the spectrum, 10% of the photons reaching the eye actually hit the retina, and the minimum flux to activate the brain response is 100 photons/s?

#5

A fluid in thermodynamic equilibrium at Temperature T fills a rigid cubical container of volume V. For wavelengths relevant to the questions below assume sound speed in the fluid (v_s) to be independent of wavelength λ (<< interatomic spacing).

1) What is the lowest angular frequency (ω_0) for a standing sound wave in the fluid?

2) What is the average energy in that mode when $k_B T \gg \hbar\omega_0$? (k_B is the Boltzmann constant; $\hbar \equiv h/2\pi$; neglect "zero point" energy)

3) What is the average energy in that mode when the inequality in 2) does not hold?

4) What is the probability for finding no energy in this mode (neglecting "zero point" energy)?

5) What is the total energy in the modes whose wavelengths lie between λ and $\lambda + d\lambda$ and are $<< V^{1/3}$?

Answers To Stat. Mech. Problem

Suggested credits for correct answer out of a perfect 10

1) $$\omega_0 = \frac{\pi}{V^{1/3}} \sqrt{3}\, v_s$$

(2/10)

1 Dim: min $kL = \pi$

$$\frac{\omega}{v_s} = k$$

3 Dim $$\omega = \sqrt{k_x^2 + k_y^2 + k_z^2}\; v_s$$

(2/10) 2) Classical regime in which

$$\bar{\varepsilon}_{\text{Harmonic oscillator}} = k_B T$$

(3/10) 4) $$P_n = \frac{e^{-n\hbar\omega_0\beta}}{\sum_{n=0}^{\infty} (e^{-\hbar\omega_0\beta})^n} \qquad \left(\beta = \frac{1}{kT}\right)$$

$$P_{n=0} = \frac{1}{\Sigma} = 1 - e^{-\hbar\omega_0\beta}$$

(3/10) 3) $$\sum n\hbar\omega_0 P_n = \frac{\hbar\omega_0}{e^{\hbar\omega_0/kT} - 1} = \langle \epsilon \rangle$$

(1/10) 5) $$\frac{4\pi\, k^2\, dk\, V\, \hbar\omega_0}{e^{\hbar k v_s \beta} - 1}$$

Thermo - Hughes
Sec. 6 General II
#6

THERMODYNAMICS

The relationship between the free energy F and the internal energy U, temperature T and entropy S of a gas with a fixed number of atoms is given by:

$$F = U - TS$$

(a) Find an expression for pressure P and entropy S expressed as a partial derivatives with respect to the free energy.

(b) Write an expression for $\left(\frac{\partial S}{\partial V}\right)_T$ in terms of pressure, volume and temperature.

(c) Use the result from part (b) to show that $\left(\frac{\partial U}{\partial T}\right)_V = -T\left(\frac{\partial P}{\partial T}\right)_V\left(\frac{\partial V}{\partial T}\right)_S$

(d) Show that $\left(\frac{\partial T}{\partial V}\right)_S = \frac{\left(\frac{\partial V}{\partial T}\right)_P\left(\frac{\partial P}{\partial V}\right)_T}{\left(\frac{\partial S}{\partial T}\right)_V}$

Thermo

(a) At constant N,

$$dU = dQ - p\,dV = T\,dS - p\,dV$$

$$dF = dU - T\,dS - S\,dT$$

$$= T\,dS - p\,dV - T\,dS - S\,dT$$

$$= -p\,dV - S\,dT$$

But $dF = \left(\frac{\partial F}{\partial V}\right)_T dV + \left(\frac{\partial F}{\partial T}\right)_V dT$

$$\boxed{\begin{aligned} p &= -\left(\frac{\partial F}{\partial V}\right)_T \\ S &= -\left(\frac{\partial F}{\partial T}\right)_V \end{aligned}}$$

(b) $$\frac{\partial^2 F}{\partial V \partial T} = \frac{\partial^2 F}{\partial T \partial V}$$

$$\left[\frac{\partial}{\partial V}\underbrace{\left(\frac{\partial F}{\partial T}\right)_V}_{-S}\right]_T = \left[\frac{\partial}{\partial T}\underbrace{\left(\frac{\partial F}{\partial V}\right)_T}_{-P}\right]_V$$

$$-\left(\frac{\partial S}{\partial V}\right)_T = -\left(\frac{\partial P}{\partial T}\right)_V$$

$$\boxed{\left(\frac{\partial S}{\partial V}\right)_T = \left(\frac{\partial P}{\partial T}\right)_V}$$

(c) $$\left(\frac{\partial U}{\partial T}\right)_V = T\left(\frac{\partial S}{\partial T}\right)_V$$ since $dU = TdS - pdV$

$$= -T\left(\frac{\partial S}{\partial V}\right)_T\left(\frac{\partial V}{\partial T}\right)_S$$

where we use : $$\left(\frac{\partial S}{\partial T}\right)_V\left(\frac{\partial T}{\partial V}\right)_S\left(\frac{\partial V}{\partial S}\right)_T = -1$$

From part b:

$$\left(\frac{\partial U}{\partial T}\right)_V = -T\left(\frac{\partial P}{\partial T}\right)_V\left(\frac{\partial V}{\partial T}\right)_S$$ ✓

(d) $\left(\frac{\partial T}{\partial V}\right)_S = -\left(\frac{\partial T}{\partial S}\right)_V \left(\frac{\partial S}{\partial V}\right)_T$

$$= -\left(\frac{\partial T}{\partial S}\right)_V \left(\frac{\partial P}{\partial T}\right)_V$$ part b

$$= \frac{-\left(\frac{\partial P}{\partial T}\right)_V}{\left(\frac{\partial S}{\partial T}\right)_V}$$

$$= \frac{\left(\frac{\partial P}{\partial V}\right)_T \left(\frac{\partial V}{\partial T}\right)_P}{\left(\frac{\partial S}{\partial T}\right)_V}$$ ✓