Columbia University
Department of Physics
QUALIFYING EXAMINATION
Friday, January 15, 2010
1:00 PM - 3:00 PM

General Physics (Part I)
Section 5.

Two hours are permitted for the completion of this section of the examination. Choose **4 problems** out of the 6 included in this section. Remember to hand in **only** the 4 problems of your choice (if by mistake you hand in 5 or 6 problems, the highest scoring problem grade(s) will be dropped). Apportion your time carefully.

Use separate answer booklet(s) for each question. Clearly mark on the answer booklet(s) which question you are answering (e.g., Section 5 (General Physics), Question 2; Section 5 (General Physics), Question 6; etc.)

Do **NOT** write your name on your answer booklets. Instead clearly indicate your **Exam Letter Code**.

You may refer to the single handwritten note sheet on 8 $1/2 \times 11$" paper (double-sided) you have prepared on General Physics. The note sheet cannot leave the exam room once the exam has begun. This note sheet must be handed in at the end of today's exam. Please include your Exam Letter Code on your note sheet. No other extraneous papers or books are premitted.

Simple calculators are permitted. However, the use of calculators for storing and/or recovering formulae or constants is NOT permitted.

Questions should be directed to the proctor.

Good luck!!

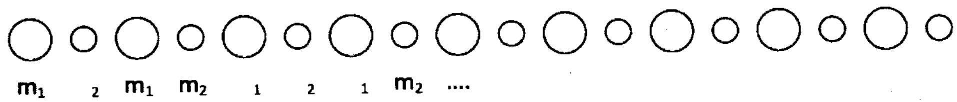
1. The Crab pulsar has a rotational period $p = 33$ msec and a period derivative of $\dot{p} = 4 \times 10^{-13}$.

   (a) Making appropriate estimates of the mass M and radius R for the Crab pulsar, estimate the current luminosity of the Crab pulsar.

   (b) The Crab pulsar emits via magnetic dipole radiation, $L_{\mathrm{m}} = \mathrm{k}\omega^4$. Assuming that the Crab pulsar was born with an initial period $p_i \ll p_{\mathrm{now}}$ ($\omega_i \gg \omega_{\mathrm{now}}$). Use the current values of p and $\dot{p}$ to estimate the age of the Crab pulsar.

   (c) Make a dimensional estimate of the luminosity of the Crab pulsar, due to magnetic dipole radiation, in terms of the magnetic field B, and other pulsar parameters. Use this estimate of the luminosity, along with the L determined in (a), to estimate the magnetic field of the Crab pulsar.

2. Imagine a one-dimensional chain of N atoms (lattice spacing ‘a’) where alternate atoms have different masses as pictured below:



Assume that the two masses are nearly equal:

$$m_1 = m(1+\Delta) \qquad\qquad m_2 = m(1-\Delta)$$

where $\Delta \ll 1$.

Solve for the normal modes (phonons) of the chain by the following steps:

(a) First solve for the case $\Delta = 0$ (equal masses) as follows. The Hamiltonian of the system is given by

$$H = \sum_n \frac{p_n^2}{2m} + \frac{1}{2} m\omega_0^2 \sum_n (x_n - x_{n+1})^2$$

Here p_n is the momentum of the n^{th} atom and x_n is its displacement from its equilibrium position $X_n = na$. The potential energy is thus determined by the relative position of the nearest neighbors. This can be solved by changing variables to Fourier space:

$$x_n = \sum_k x_k e^{ikna} \quad \text{and} \quad p_n = \sum_k p_k e^{-ikna}$$

with $k = 0, \pm\frac{\pi}{Na}, \pm\frac{2\pi}{Na}, \cdots, \pm\frac{\pi}{a}$.

Show that in Fourier space the Hamiltonian reduces to

$$H = \sum_k \frac{p_k p_{-k}}{2m} + \frac{1}{2} m\omega_k^2 \sum_k x_k x_{-k}$$

Find ω_k, the dispersion relation between the energy ω and momentum k. Sketch your result. What is this kind of phonon called?

(b) Now solve for the case of unequal masses by expanding the Hamiltonian to first order in Δ. The zeroth order in Δ results in the phonon mode you have found in part (a). What is the first order Hamiltonian?

(c) Solve the first order Hamiltonian you found in (b) in exactly the same way as you did in (a). What is the new dispersion relation? Sketch the results for the two modes. What is the new phonon mode called?

3. In many experiments, the surface of the sample or detector being used has to be placed in a vacuum environment to avoid contamination from air molecules.

   (a) Estimate the pressure in a vacuum chamber (in atmospheres) where one air molecule hits every surface atom of the walls of the chamber every second. Assume that air is composed of only nitrogen molecules (molecular weight 28) that travel at 500 m/s. Assume also that a typical atom on the wall of the chamber has a size of 1 Angstrom (1 atm = 1.013×10^5 Pa).

   (b) Such low pressures are reached by the use of vacuum pumps. A vacuum pump operates by displacing a certain volume C per second from the chamber which is then exhausted externally (imagine a chamber where the volume of the chamber is continuously increased by C per second, resulting in a continuous drop in pressure). How long will it take a vacuum pump with a displacement of 1 liter per second to reduce the pressure in a 100 liter chamber from atmosphere to the pressure required in (a)? Assume that temperature is held constant throughout.

4. A perfect fluid is described by the continuity and Euler equations, which govern the time-evolution of the density and velocity fields $\rho(\vec{x},t)$, $\vec{v}(\vec{x},t)$:

$$\dot{\rho} + \vec{\nabla}\cdot(\rho\vec{v}) = 0$$
$$\dot{\vec{v}} + (\vec{v}\cdot\vec{\nabla})\vec{v} = -\frac{1}{\rho}\vec{\nabla}p.$$

Here we assume that the pressure p is a given function of ρ:

$$p = p(\rho).$$

(a) Linearize the equations of motion above, for small fluctuations $\delta\rho$ and $\delta\vec{v}$, about the homogeneous, static background configuration

$$\rho = \rho_0\,, \qquad \vec{v} = 0\,.$$

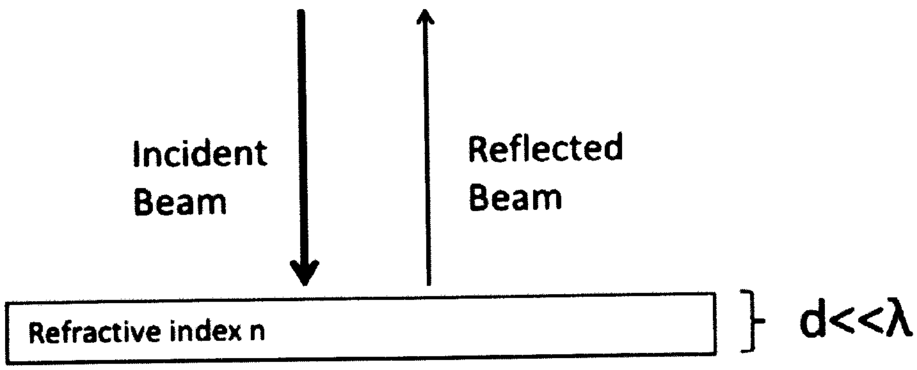
(b) Consider plane wave-like configurations for $\delta\rho$ and $\delta\vec{v}$:

$$\delta\rho(\vec{x},t) = \delta\rho_*(t)e^{i\vec{k}\cdot\vec{x}} + \text{c.c}\,, \qquad \delta\vec{v}(\vec{x},t) = \delta\vec{v}_*(t)e^{i\vec{k}\cdot\vec{x}} + \text{c.c}$$

Solve the linear equations you derived in part (a) for $\delta\rho_*(t)$ and $\delta\vec{v}_*(t)$.
Hint: decompose $\delta\vec{v}$ into transverse and longitudinal parts.

(c) What do these solutions describe physically?

5. We wish to detect the presence of a thin membrane suspended in vacuum by reflection of a light beam impinging at normal incidence. Model the material as a thin slab of homogeneous, transparent material with a refractive index n and a thickness d.



(a) Find an *explicit* expression for the reflectance R of the slab in the limit of $d \ll \lambda$, where λ is the vacuum wavelength of light.

(b) Estimate the minimum effective thickness of a membrane that could, in principle, be detected in this fashion. Assume typical parameters for a dielectric material, that we have available a 1 μW visible laser, and that we are able to detect $10^9/s$ photons of reflected light. Use the relation derived above or, if unavailable, a suitable approximate expression.

6. Consider the rotational degree of freedom of a dilute gas of diatomic CO molecules at temperature T. Suppose that the moment of inertia of each molecule is I.

   (a) Write an explicit expression for the (quantum) partition function Z_{rot} for the rotational degree of freedom of one molecule. Although you may not be able to reduce it to closed form, make sure that all quantities in Z_{rot} are defined so that it could be evaluated numerically.

   (b) Write a general expression for the heat capacity per molecule associated with rotational motion in terms of Z_{rot}.

   (c) Obtain an analytic expression for the asymptotic behavior of the rotational contribution to the heat capacity per molecule in the limit of low temperature.

   (d) For CO molecules, approximately how low does the temperature have to be so that the relation derived in part (c) is applicable. Use suitable estimates of the relevant physical parameters.

Astro Hailey
Sect 5
General I

Hailey : general : astrophysics

#1

The Crab pulsar has a rotational period $P = 33$ msec And a period derivative of $\dot{P} = 4 \times 10^{-13}$ s/s

a.) Making appropriate estimates of M And R for the Crab pulsar, estimate the current luminosity of the Crab pulsar.

b.) The Crab pulsar emits via magnetic dipole radiation, $L_m \propto K\omega^4$. Assuming that the Crab pulsar was born with an initial period $P_i << P_{now}$ $(\omega_i >> \omega_{now})$, use the current values of P And $\dot{P}$ to estimate the age of the Crab pulsar.

c.) Make a dimensional estimate of the luminosity of the Crab pulsar ~~in terms~~ due to magnetic dipole radiation in terms of B And other pulsar parameters. Use your estimate of L from (a) along with the L determined in (c) to estimate the magnetic field of the Crab pulsar.

Solution : Hailey : Astrophysics : general

a.) $\dot{E} = \frac{d}{dt}(\tfrac{1}{2} I\omega^2) = I\omega\dot{\omega}$

$\omega = 2\pi/P \;;\; \dot{\omega} = \frac{2\pi}{P^2}\dot{P}$

$\omega \simeq 200 \frac{rad}{s} \quad \dot{\omega} \simeq 2\times10^{-9}$ rad/s

This is a neutron star so $M \simeq 1 M_\odot \sim 2\times10^{33}$ g

$R \simeq 10$ km $\simeq 10^6$ cm

$I \sim (2/5) MR^2 \simeq 1.4\times10^{45}$ g-cm^2

$\dot{E} \simeq 6\times10^{38}$ erg/s Ans

b.) $L = I\omega\dot{\omega} = -K\omega^4$

$$\int_{\omega_i}^{\omega} \frac{d\omega}{\omega^3} = \int -\frac{dt\,K}{I}$$

$$-Kt/I = -\left(\frac{1}{\omega^2} - \frac{1}{\omega_i^2}\right)\tfrac{1}{2} \qquad \omega_i >> \omega$$

$$\omega = \sqrt{\frac{I}{K}}\, t^{-1/2} \;;\; \dot{\omega} = \sqrt{\frac{I}{K}}\,\frac{1}{2}\, t^{-3/2}$$

$$\dot{\omega}/\omega = \frac{1}{2t} \qquad t = \frac{1}{2(\dot{\omega}/\omega)} \simeq 1600 \text{ yrs}$$

Ans

c.) Magnetic dipole radiation $L_m \sim \frac{(\ddot{M})^2}{c^3}$

$M_0 \sim \frac{IA}{c} \qquad \int B\,dl \sim BR \simeq \frac{4\pi}{c} I$

$M_0 \sim \frac{B}{4\pi} 4\pi R^2 R \qquad \frac{(\ddot{M})^2}{c^3} \simeq \omega^4 \frac{(BR^3)^2}{c^3}$ →

$$L_m \simeq \frac{B^2 R^6 \omega^4}{c^3}$$

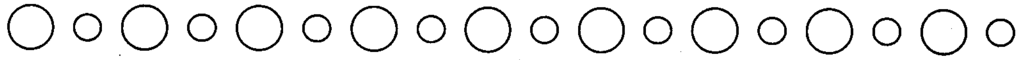
From (a) $L \simeq 6\times10^{38} \sim \frac{B^2 R^6}{c^3}\omega^4$

Solving $B \simeq 2\times10^{12}$ gauss Ans 2

CM - Pasupathy
Sec. 5 general I
2
condensed matter

Condensed Matter

Imagine a one-dimensional chain of atoms where alternate atoms have different masses as pictured below:



Assume that the two masses are nearly equal:

$$m_1 = m(1 + \Delta)$$

$$m_2 = m(1 - \Delta)$$

where $\Delta \ll 1$.

Solve for the normal modes (phonons) of the chain by the following steps:

1. First solve for the case $\Delta = 0$ (equal masses) as follows. The Hamiltonian of the system is given by

$$H = \sum_n \frac{p_i^2}{2m} + \frac{1}{2} m\omega_0^2 \sum_n (x_n - x_{n+1})^2$$

here p_n and x_n are the momentum and position of the n^{th} atom in the chain. The potential energy is thus determined by the relative position of the nearest neighbors. This can be solved by changing variables to Fourier space

$$x_n = \sum_k x_k e^{ikn}$$

$$p_n = \sum_k p_k e^{-ikn}$$

Show that in Fourier space the Hamiltonian reduces to

$$H = \sum_n \frac{p_k p_{-k}}{2m} + \frac{1}{2} m\omega_k^2 \sum_n x_k x_{-k}$$

Find ω_k, the relationship (dispersion) between the energy ω and momentum k. Sketch your result. What is this kind of phonon called?

2. Now solve for the case of unequal masses by expanding the Hamiltonian to the first power of Δ. The zeroth order in Δ results in the phonon mode you have found in (1). What is the first order Hamiltonian?

3. Solve this first order Hamiltonian exactly as in (1). What is the new dispersion relation? Sketch the results for the two modes. What is the new phonon mode called?

Solution

$$H = \sum_n \frac{p_i^2}{2m} + \frac{1}{2} m\omega_0^2 \sum_n (x_n - x_{n+1})^2$$

use

$$x_n = \sum_k x_k e^{ikn}$$

$$p_n = \sum_k p_k e^{-ikn}$$

$$H = \sum_{n,k,k'} \frac{p_k e^{-ikn} p_{k'} e^{-ik'n}}{2m} + \frac{1}{2} m\omega_0^2 \sum_n \left(\sum_k x_k e^{ikn} - \sum_k x_k e^{ik(n+1)} \right)^2$$

Use the fact that

$$\sum_n e^{-ikn} e^{-ik'n} = \delta(k + k')$$

to simplify:

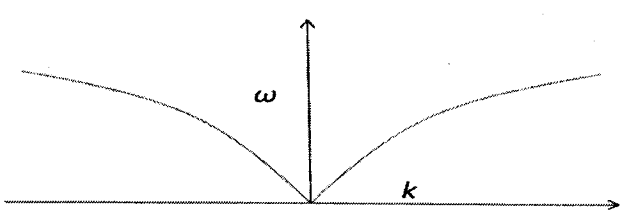
$$H = \sum_k \frac{p_k p_{-k}}{2m} + \frac{1}{2} m\omega_0^2 \sum_{n,k,k'} 2\left(x_k e^{ikn} x_{k'} e^{ik'n} - x_k e^{ikn} x_{k'} e^{ik'(n+1)}\right)$$

$$= \sum_k \frac{p_k p_{-k}}{2m} + \frac{1}{2} m\omega_0^2 \sum_{n,k,k'} 2x_k x_{-k} (1 - \cos(k))$$

$$= \sum_k \frac{p_k p_{-k}}{2m} + \frac{1}{2} m\omega_k^2 \sum_k x_k x_{-k}$$

where

$$\omega_k = \omega_0 \sqrt{(1 - \cos(k))} = \omega_0 \left| \sin\left(\frac{k}{2}\right) \right|$$



ACOUSTIC MODE

(2)

$$H = \sum_{odd\ n} \frac{p_i^2}{2m(1+\Delta)} + \frac{1}{2}m(1+\Delta)\omega_0^2 \sum_{odd\ n} (x_n - x_{n+1})^2$$
$$+ \sum_{even\ n} \frac{p_i^2}{2m(1-\Delta)} + \frac{1}{2}m(1-\Delta)\omega_0^2 \sum_{even\ n} (x_n - x_{n+1})^2$$
$$= \sum_n \frac{p_i^2}{2m}(1+(-1)^n\Delta) + \frac{1}{2}m\omega_0^2 \sum_{odd\ n} (x_n - x_{n+1})^2(1+(-1)^n\Delta)$$
$$= H_0 + H_\Delta$$

Here H_0 is the zeroth order Hamiltonian and

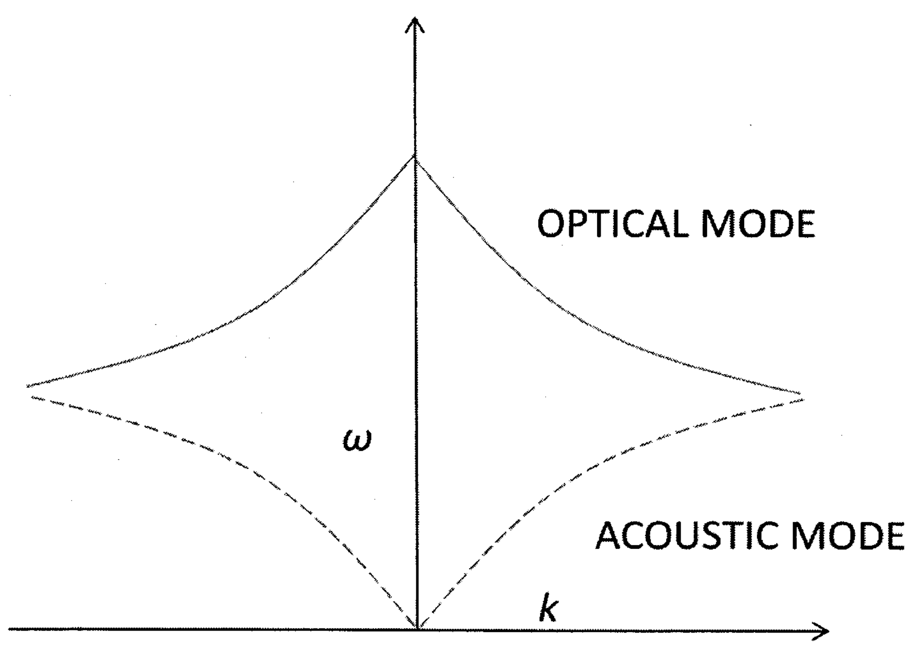
$$H_\Delta = \sum_n \frac{p_i^2}{2m}(-1)^{n+1}\Delta + \frac{1}{2}m\omega_0^2 \sum_n (x_n - x_{n+1})^2(-1)^n\Delta$$

(3) Once again, Fourier transform and use $-1 = e^{i\pi}$

$$H_\Delta = \sum_k \frac{p_k p_{\pi-k}}{2m} + \frac{1}{2}m\omega_0^2 \sum_k 2x_k x_{\pi-k}(1+\cos(k))$$

The new mode is therefore given by

$$\omega_k = \omega_0\sqrt{(1+\cos(k))} = \omega_0 \left|\cos(\frac{k}{2})\right|$$



Expt Techniques - Pasupathy
Sec. 5 General I
3

General Experiment

In many experiments, the surface of the sample or detector being used has to be placed in a vacuum environment to avoid contamination from air molecules.

(a) Estimate the pressure in a vacuum chamber (in atmospheres) where one air molecule hits every surface atom of the walls of the chamber every second. Assume that air is composed of only nitrogen molecules (molecular weight 28) that travel at 500 m/s. Assume also that a typical atom on the wall of the chamber has a size of 1 Angstrom.

(b) Such low pressures are reached by the use of vacuum pumps. A vacuum pump operates by displacing a certain volume C per second from the chamber which is then exhausted externally (imagine a chamber where the volume of the chamber is continuously increased by C per second resulting in a continuous drop in pressure). How long will it take a vacuum pump with a displacement of 1 liter per second to reduce the pressure in a 100 liter chamber from atmosphere to the pressure required in (a)? Assume that temperature is maintained constant throughout.

Solution

(a) Momentum of nitrogen molecule = $\frac{M}{N}v$

where M= 28/1000 kg, N = 6 X 10^{23}, v = 500 m/s

If one molecule bounces off the surface atom per second, the net force per second is $\frac{2M}{N}v$ and pressure

$$P = \frac{2Mv}{NA}$$

$$P = \frac{2 * 0.028 * 500}{6 * 10^{23} * 10^{-20}} = 5 * 10^{-3} Pa = 5 * 10^{-8}\, Atmospheres$$

(b) Assume the chamber has a volume V_0. The rate of change of the number of molecules in the chamber dN is given by (assuming the pump is displacing air out of the chamber)

$$\frac{dN}{N} = -\frac{dV}{V_0} = -\frac{1}{V_0}\frac{dV}{dt}dt = -\frac{C}{V_0}dt$$

At constant temperature the number of molecules in the chamber is proportional to the pressure, so

$$\frac{dP}{P} = -\frac{C}{V_0}dt$$

Therefore

$$P(t) = P_0 e^{-\frac{C}{V_0}t}$$

Plug in the numbers

$$\frac{t}{100} = \ln(5 * 10^{8}) = 2000\, seconds$$

Fluids Nicolis
Sec 5
General I
4 (fluids)

Quals 2009-10

1 General: Fluids – spectrum of small fluctuations

A perfect fluid is described by the continuity and Euler equations, which govern the time-evolution of the density- and velocity-fields $\rho(\vec{x},t)$, $\vec{v}(\vec{x},t)$:

$$\dot{\rho} + \vec{\nabla}\cdot(\rho\vec{v}) = 0 \tag{1}$$

$$\dot{\vec{v}} + (\vec{v}\cdot\vec{\nabla})\vec{v} = -\frac{1}{\rho}\vec{\nabla}p\,. \tag{2}$$

Here we assume that the pressure p is a given function of ρ:

$$p = p(\rho)\,. \tag{3}$$

1. Linearize the equations of motion above, for small fluctuations $\delta\rho$, $\delta\vec{v}$ about the homogeneous, static background configuration

$$\rho = \rho_0\,, \qquad \vec{v} = 0\,. \tag{4}$$

2. Consider plane wave-like configurations for $\delta\rho$ and $\delta\vec{v}$:

$$\delta\rho(\vec{x},t) = \delta\rho_*(t)\,e^{i\vec{k}\cdot\vec{x}} + \text{c.c.}\,, \qquad \delta\vec{v}(\vec{x},t) = \delta\vec{v}_*(t)\,e^{i\vec{k}\cdot\vec{x}} + \text{c.c.} \tag{5}$$

Solve the linear equations you derived in item 1 for $\delta\rho_*(t)$ and $\delta\vec{v}_*(t)$. Hint: decompose $\delta\vec{v}$ into a transverse part and a longitudinal one.

3. What do these solutions describe, physically?

Solution

1. At linear order in $\delta\rho$, $\delta\vec{v}$, eqs. (1,2) reduce to

$$\delta\dot{\rho} + \rho_0\vec{\nabla}\cdot\delta\vec{v} = 0 \tag{6}$$

$$\delta\dot{\vec{v}} + \frac{1}{\rho_0}c_s^2\,\vec{\nabla}\delta\rho = 0\,, \tag{7}$$

where we used that the background has $\vec{v}=0$, that p is a function of ρ, so that

$$\vec{\nabla}p = \frac{dp}{d\rho}\vec{\nabla}\rho\,, \tag{8}$$

and we defined c_s^2 as

$$c_s^2 = \left.\frac{dp}{d\rho}\right|_{\rho_0}\,. \tag{9}$$

2. For configurations of the form (5) the linearized equations reduce to

$$\delta\dot{\rho}_* + \rho_0\, i\vec{k}\cdot\delta\vec{v}_* = 0 \qquad (10)$$

$$\delta\dot{\vec{v}}_* + \frac{1}{\rho_0}\, c_s^2\, i\vec{k}\,\delta\rho_* = 0\,. \qquad (11)$$

We now project $\delta\vec{v}_*$ and the second equation onto the parallel and transverse (w.r.t. $\vec{k}$) directions. We get two coupled equations for $\delta\rho$ and $\delta v_*^{\parallel}$

$$\delta\dot{\rho}_* + \rho_0\, ik\,\delta v_*^{\parallel} = 0 \qquad (12)$$

$$\delta\dot{v}_*^{\parallel} + \frac{1}{\rho_0}\, c_s^2\, ik\,\delta\rho_* = 0\,, \qquad (13)$$

and a trivial equation for $\delta\vec{v}_*^{\perp}$:

$$\delta\dot{\vec{v}}_*^{\perp} = 0 \qquad (14)$$

By using either of eqs. (12, 13) in the other, one gets an ordinary wave equation for $\delta\rho$ and $\delta v_*^{\parallel}$, with solutions

$$\delta\rho_*(t) = \delta\bar{\rho}\, e^{-i\omega t} \qquad \delta v_*^{\parallel} = \frac{\delta\bar{\rho}}{\rho_0} c_s\, e^{-i\omega t} \qquad \omega \equiv c_s k\,. \qquad (15)$$

The relative phase and amplitude are fixed by either of eqs. (12, 13).

The solution to eq. (14) is instead

$$\delta\vec{v}_*^{\perp} = \text{const}\,. \qquad (16)$$

3. The oscillatory solutions (15) obviously describe sound waves: they are longitudinal ($\delta\vec{v} \parallel \vec{k}$) compressional ($\delta\rho \neq 0$) modes. The transverse fluctuations instead describe vortices. More precisely, the linearized version thereof. Indeed in real space transversality means

$$\vec{\nabla}\cdot\delta\vec{v}^{\perp} = 0\,, \qquad (17)$$

which implies that $\delta\vec{v}^{\perp}$ is a curl:

$$\delta\vec{v}^{\perp} = \vec{\nabla}\times\vec{A}\,. \qquad (18)$$

The trivial dynamics (16) matches the fact that a vortex in constant rotation is a solution.

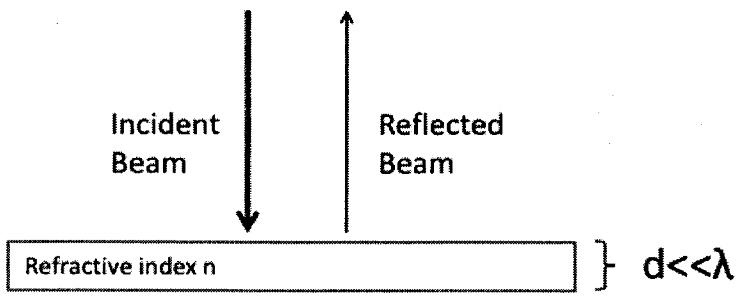
2 General: Astrophysics – the Hubble flow

Consider a self-gravitating, infinitely extended fluid. Assume that the fluid has negligible pressure. The relevant equations are the continuity and Euler ones for the fluid's dynamics, and the Poisson equation for the Newtonian potential. Call ρ the fluid's density, $\vec{v}$ its velocity field, and Φ the gravitational potential per unit mass.

Optics Hinz
Sec. 5. General I
5 (Optics)

OPTICS PROBLEM

We wish to detect the presence of a thin membrane suspended in vacuum by reflection of a light beam impinging at normal incidence. Model the material as a thin slab of homogeneous, transparent material with a refractive index *n* and a thickness *d.*



(a) Find an *explicit* expression for the reflectance *R* of the slab in the limit of $d \ll \lambda$, where λ is the vacuum wavelength of light.

(b) Estimate the minimum effective thickness of a membrane that could in principle be detected in this fashion. Assume typical parameters for a dielectric material; that we have available a 1 μW visible laser and; that we are able to detect 10^9/s photons of reflected light. Use the relation derived above or, if unavailable, a suitable approximate expression.

<u>Solution</u>: We can analyze the problem either

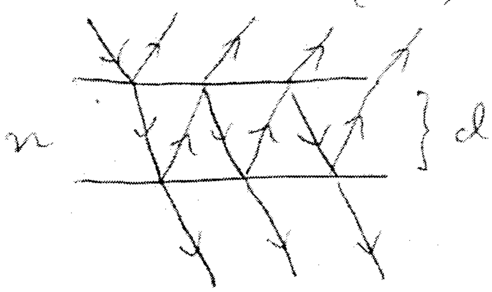
(a) by a direct solution of the boundary value problem or using multiple reflections. Here we present the latter. We use reflection and transmission coefficients for the electric field for normal incidence radiation for a boundary of $n_1 \to n_2$ of

$$r_{12} = (n_1 - n_2)/(n_1 + n_2) \qquad t_{12} = 2n_1/(n_1+n_2)$$

For our case

$r = (1-n)/(1+n) \qquad t = 2/(1+n)$ entering slab

$r' = (n-1)/(1+n) \qquad t' = 2n/(1+n)$ exiting slab



phase shift in propagation
$e^{i\delta} = e^{i 2\pi n d/\lambda}$

$$r_{tot} = r + tt'r'e^{2i\delta}\left[1 + r'^2 e^{2i\delta} + (r'^2 e^{2i\delta})^2 + \dots\right]$$

$$= r + tt'r'e^{2i\delta}/(1 - r'^2 e^{2i\delta}) = r\left[1 - \frac{nt^2 e^{2i\delta}}{1-r^2 e^{2i\delta}}\right]$$

$$= r\left[1 - e^{2i\delta}(r^2 + nt^2)\right]/\left[1 - r^2 e^{2i\delta}\right]$$

$$= \left[r/(1-r^2)\right](2i\delta) \text{ to leading order in } \delta \ll 1$$

$$= \left[(1-n^2)/4n\right](4\pi n d/\lambda) = i\frac{\pi d}{\lambda}(1-n^2) = i\frac{\pi d}{\lambda}\chi \leftarrow \text{suscept.}$$

The power reflection coefficient is

$$R_{tot} = |r_{tot}|^2 = (\pi d/\lambda)^2(1-n^2) \quad \|$$

(b) From the above, we see that for a given R_{tot} that we can measure, we have

$$d = \frac{\lambda R_{tot}^{1/2}}{\pi(n^2-1)}$$

From the given data, we can determine R_{tot}. In the visible, a typical photon energy is 1 eV, so $1\,\mu W = 10^{-6}\, J/s \leftrightarrow 10^{13}$ photons/s. If we can detect $10^9/s$, then we can measure $R_{tot} = 10^{-4}$.

For a typical $n^2 \sim 3$ and $\lambda \sim 1\,\mu m$, we obtain

$$d = 10^{-6} m\, 10^{-2}/2\pi \sim 1\, nm$$ ||

Stat Mech. Heinz
Sec. 5 General I
\# 6

STATISTICAL MECHANICS PROBLEM

Consider the rotational degree of freedom of a dilute gas of diatomic CO molecules at temperature T. Suppose that the moment of inertia of each molecule is I.

(a) Write an explicit expression for the (quantum) partition function Z_{rot} for the rotational degree of freedom of one molecule. Although you may not be able to reduce to closed form, make sure that all quantities in Z_{rot} are defined so that it could be evaluated numerically.

(b) Write in terms of Z_{rot} a general expression for the heat capacity per molecule associated with rotational motion.

(c) Obtain an analytic expression for the asymptotic behavior of the rotational contribution to the heat capacity per molecule in the limit of low temperature.

(d) For CO molecules, approximately how low does the temperature have to be so that the relation derived in part (c) is applicable. Use suitable estimates of the relevant physical parameters.

Solution

(a) The rotational spectrum follows from $H = J^2/2I$, where I is the moment of inertia of the molecule. This yields energy levels $E_j = j(j+1)\hbar^2/2I$ with $j = 0, 1, 2, 3, \ldots$ Each level has a degeneracy of $(2j+1)$, corresponding to the allowed values of m_j

$$\therefore \quad Z_{rot} = \sum_{j=0}^{\infty} (2j+1)\, e^{-j(j+1)\hbar^2/2I\,\beta} \quad \|$$

with $\beta = (kT)^{-1}$ being the inverse temperature

(b) $\bar{E}_{rot} = -\frac{\partial}{\partial\beta} \ln Z_{rot}$ and $C_{rot} = \frac{\partial E_{rot}}{\partial T} = -k\beta^2 \frac{\partial E_{rot}}{\partial \beta}$

$$C_{rot} = k\beta^2 \frac{\partial^2}{\partial\beta^2} \ln Z_{rot} \quad \|$$

$$\Gamma = k\left(T^2 \frac{\partial^2}{\partial T^2} + 2T\frac{\partial}{\partial T}\right) \ln Z_{rot}$$

(c) The asymptotic behavior for low T (high β) is found by keeping the leading-order T-dep. term:

$$Z_{rot} = 1 + 3e^{-2\epsilon\beta} \quad \text{with } \epsilon = \hbar^2/2I$$

Keeping only the leading-order terms (slowest decay in T):

$$C_{rot} = 12k\left(\epsilon/kT\right)^2 e^{-2\epsilon/kT} \quad \|$$

(d) The low-temperature limit is valid for
$kT \ll \epsilon = \hbar^2/2I = h^2/8\pi^2 I$

For a diatomic molecule $I = \mu R^2$ where $\mu = (m_1 m_2)/(m_1+m_2)$ is the reduced mass and R is the bond length. For CO, $R = 0.11\,nm$ and $\mu = 12 \cdot 16/(12+16)$ amu $= 7$ amu. Then

$$T \ll \frac{\epsilon}{k} = \frac{h^2}{8\pi^2 k \mu R^2} = \frac{(hc)^2}{8\pi k \mu c^2 R^2}$$

$$= \frac{(1240\ eV\text{-}nm)^2}{8\pi^2 (8.6\times10^{-5}\ eV/K)(7)(930\ MeV)(0.11\ nm)^2}$$

$$= 2.9\,K \;\; ||$$