Columbia University
Department of Physics
QUALIFYING EXAMINATION
Wednesday, January 13, 2010
1:00 PM - 3:00 PM

Quantum Mechanics
Section 3.

Two hours are permitted for the completion of this section of the examination. Choose **4 problems** out of the 5 included in this section. Remember to hand in **only** the 4 problems of your choice (if by mistake you hand in 5 problems, the highest scoring problem grade will be dropped). Apportion your time carefully.

Use separate answer booklet(s) for each question. Clearly mark on the answer booklet(s) which question you are answering (e.g., Section 3 (Quantum Mechanics), Question 2; Section 3 (Quantum Mechanics), Question 6; etc.)

Do **NOT** write your name on your answer booklets. Instead clearly indicate your **Exam Letter Code**.

You may refer to the single handwritten note sheet on 8 $1/2 \times 11$" paper (double-sided) you have prepared on Quantum Mechanics. The note sheet cannot leave the exam room once the exam has begun. This note sheet must be handed in at the end of today's exam. Please include your Exam Letter Code on your note sheet. No other extraneous papers or books are premitted.

Simple calculators are permitted. However, the use of calculators for storing and/or recovering formulae or constants is NOT permitted.

Questions should be directed to the proctor.

Good luck!!

1. Two observers in different inertial frames will need different wave functions to describe the same physical system. To make things simple, consider how it works in the non-relativistic case. The first observer uses coordinates $(\vec{x}, t)$ and a wave function $\psi(\vec{x}, t)$, while the second uses $(\vec{x}\,', t)$ and $\psi(\vec{x}\,', t)$. Of course, $\vec{x}\,' = \vec{x} - \vec{v}t$ where $\vec{v}$ is a constant velocity. The wave functions for the two observers are said to be related as follows:

$$\tilde{\psi}(\vec{x}\,', t) = \psi(\vec{x}, t) \exp\left(\frac{-i}{\hbar}\left[m\vec{v}\cdot\vec{x} - \frac{mv^2}{2}t\right]\right)$$

Despite its innocuous look (it's just a phase!) this transformation has interesting effects.

(a) Let us first verify that it makes sense. Suppose $\psi(\vec{x}, t)$ is the wave function of a free particle of momentum $\vec{p} = (p_x, p_y, p_z)$. Show that $\tilde{\psi}(\vec{x}\,', t)$ indeed describes a free particle with a proper momentum.

(b) Now let us put this to work. Suppose we have a hydrogen atom, which at $t < 0$ was at rest with the electron in the ground *1s* state described by the wave function

$$\psi(\vec{x}) = \psi_{1,0}(\vec{x}) \equiv \frac{1}{\sqrt{\pi a_B^3}} \exp\left(\frac{-r}{a_B}\right); \quad r = |\vec{x}|$$

where a_B is the Bohr radius.

Suppose at $t = 0$ the proton suddenly starts to move (e.g., due to a collision with a neutron) in the z-direction with the velocity v. Let the change in the velocity be so abrupt that the electron wave function remains the same. What is the probability at $t > 0$ to find the moving hydrogen atom with the electron in the ground state?

(c) What is the probability to find the electron in the state with $n = 2, l = 1, m = 1$?

2. (a) Prove that the expectation value of the Hamiltonian $E[\phi]$ is stationary in the neighborhood of a discrete eigenstate i.e., if $H\psi_n = E_n\psi_n$ and $\psi = \psi_n + \delta\psi$, then $\delta\langle\psi|H|\psi\rangle = 0$. Show also that $E[\phi] \geq E_0$, where $E_0 \leq E_n$ is the ground state energy.

   (b) Apply the above to estimate the quantum ground state energy of a simple harmonic oscillator using a trial wave function of the form $\psi(x) = \exp(-x^2/a^2)$. Determine a, and compare $E[\psi]$ to the exact E_0 ground state energy.

(Useful integrals are $\int_{-\infty}^{\infty} dx\, e^{-b^2x^2} = \sqrt{\pi}/b$ and its derivative with respect to b.)

3. Consider an electron of charge e and mass m confined on a ring of radius R. In cylindrical coordinates the Hamiltonian of this confined system can be described by

$$H = \frac{1}{2m}\left(\frac{\hbar}{i}\vec{\nabla}\right)^2 = -\frac{\hbar^2}{2m}\left(\frac{1}{R}\frac{d}{d\phi}\right)^2.$$

where ϕ is the azimuthal angle.

(a) Find the energy eigenvalues and normalized eigenfunctions of this system.

(b) Now we consider a magnetic field $\vec{\mathbf{B}} = B\hat{z}$ applied along the z-direction. Employing the "symmetric" gauge, the corresponding vector potential on the ring can be expressed by

$$\vec{\mathbf{A}} = \frac{BR}{2}\hat{\phi},$$

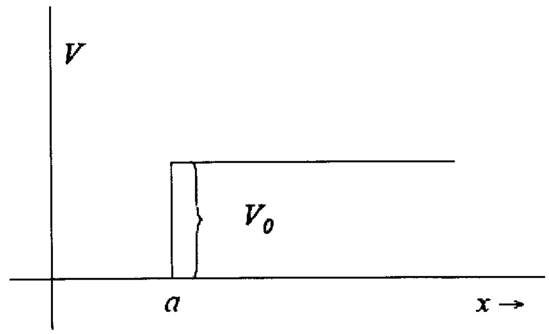
where $\hat{\phi}$ is the unit vector along the azimuthal angle ϕ. In the magnetic field, the Hamiltonian is given by $H = \frac{1}{2m}\left(\frac{\hbar}{i}\vec{\nabla} - e\vec{\mathbf{A}}\right)^2$. Find the energy eigenvalue of an electron confined to this ring in the presence of a fixed magnetic field.

(c) Find the smallest magnetic field for which one can find the non-degenerate ground and doubly degenerate excited states.

4. Consider a particle of mass m in a one-dimensional potential $V(x)$ where

$$V(x) = \infty \qquad x < 0$$
$$V(x) = 0 \qquad 0 < x < a$$
$$V(x) = V_0 \qquad x > a$$

with $V_0 > 0$.

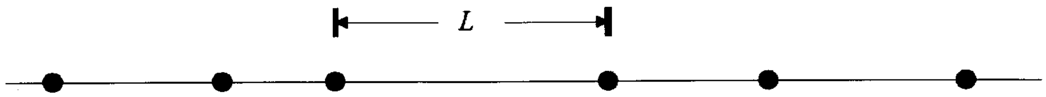


(a) If $E = \frac{\hbar^2 k^2}{2m}$ is a bound state energy and $V_0 - E = \frac{\hbar^2 \alpha^2}{2m}$, give the equation determining possible values of E.

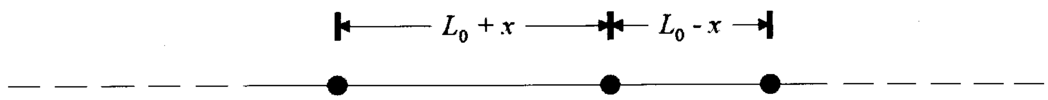
(b) Give the condition on V_0 and a for at least one bound state to exist.

(c) What are the energy levels when $V_0 = \infty$?

5. Consider a quantum system with an infinite set of particles in one dimension as shown in the figure. Particles cannot cross neighbors. We are interested in the probability distribution of spacings L between a particle and its neighbor to the left, given that the average spacing between particles is L_0.



In the simplest approximation to this many-body problem, a single particle moves between two fixed neighbors separated by $2L_0$. Let $x = L - L_0$ denote the deviation from the midpoint.



(a) Find the probability distribution $P(L)$ in the ground state, in the above approximation, and assuming there are no interparticle interactions (other than contact interactions).

(b) Now suppose there are strong repulsive potentials between pairs of neighboring particles of the form AL^{-n}. Considering only the nearest neighbors, write the potential energy for the middle particle for $x \ll L_0$. Write the result explicitly in terms of A, n and L_0.

(c) Still assuming that $x \ll L_0$ and that the neighboring particles are fixed, write the Schrödinger equation for the middle particle, and argue that the problem can be mapped into a familiar one. Based on this analogy, what is the form of the distribution $P(L)$? How does its *width* scale with L_0?

(d) In the limit of strong repulsion, as in *b*) and *c*) above, explain how you can measure the power n and the amplitude A that characterize the potential.

Altshuler
sec. 3 QM
1

Two observers in different inertial frames will need different wave functions to describe the same physical system. To make things simple consider how it works in the non-relativistic case. The first observer uses coordinates $(\vec{x},t)$ and a wave function $\psi(\vec{x},t)$, while the second uses $(\vec{x}',t)$ and $\tilde{\psi}(\vec{x}',t)$. Of course, $\vec{x}'=\vec{x}-\vec{v}t$ where $\vec{v}$ is a constant velocity. The wave functions for the two observers are said to be related as follows:

$$\tilde{\psi}(\vec{x}',t)=\psi(\vec{x},t)\exp\left(\frac{-i}{\hbar}\left[m\vec{v}\vec{x}-\frac{mv^2}{2}t\right]\right)$$

Despite its innocuous look (it's just a phase!) this transformation has interesting effects.

a) Let us first verify that it makes sense. Suppose $\psi(\vec{x},t)$ is the wave function of a free particle of momentum $\vec{p}=(p_x,p_y,p_z)$. Show that $\tilde{\psi}(\vec{x}',t)$ indeed describes a free particle with a proper momentum.

Solution:
Wave function of the free particle of momentum $\vec{p}=(p_x,p_y,p_z)$ can be written as

$$\psi(\vec{x},t)=C\exp\left(\frac{i}{\hbar}\left[\vec{p}\vec{x}-\frac{p^2}{2m}t\right]\right)=C\exp\left(\frac{i}{\hbar}\left[p_xx+p_yy+p_zz-\frac{p_x^{\,2}+p_y^{\,2}+p_z^{\,2}}{2m}\right]\right)$$

where C is the normalization constant.

$$\tilde{\psi}(\vec{x}',t)=C\exp\left(\frac{i}{\hbar}\left[\vec{p}\vec{x}-\frac{p^2}{2m}t-m\vec{v}\vec{x}+\frac{mv^2}{2}t\right]\right)$$

Now we can substitute $\vec{x}=\vec{x}'+\vec{v}t$

$$\tilde{\psi}(\vec{x}',t)=C\exp\left(\frac{i}{\hbar}\left[\vec{p}(\vec{x}'-\vec{v}t)-\frac{p^2}{2m}t+m\vec{v}(\vec{x}'+\vec{v}t)-\frac{mv^2}{2}t\right]\right)=$$

$$=C\exp\left(\frac{i}{\hbar}\left[(\vec{p}-mv)\vec{x}'+t\left(\vec{p}\vec{v}-\frac{p^2}{2m}+mv^2-\frac{mv^2}{2}\right)\right]\right)$$

$$=C\exp\left(\frac{i}{\hbar}\left[(\vec{p}-m\vec{v})\vec{x}'-t\frac{1}{2m}(p^2-2m\vec{p}\vec{v}+m^2v^2)\right]\right)=\exp\left(\frac{-i}{\hbar}\left[(\vec{p}-m\vec{v})-\frac{1}{2m}(p-m\vec{v})^2\right]\right)$$

One can see that the wave function in a new frame can be written as

$$\tilde{\psi}(\vec{x}',t)=C\exp\left(\frac{i}{\hbar}\left[\vec{p}'\vec{x}'-\frac{(p')^2}{2m}t\right]\right)$$

where $\vec{p}'\equiv\vec{p}-m\vec{v}$ .is the momentum in the new frame. It is indeed the wave function of the free particle in the new frame.

b) Now let us put this to work. Suppose we have a hydrogen atom, which at $t<0$ was at rest with the electron in the ground *1s* state described by the wave function

$$\psi(\vec{x})=\psi_{1,0}(\vec{x})\equiv\frac{1}{\sqrt{\pi a_B^3}}\exp\left(\frac{-r}{a_B}\right);\quad r=|\vec{x}|$$

Suppose at $t=0$ the proton suddenly starts to move (e.g., due to a collision with a neutron) in the *z*-direction with the velocity *v*. Let the change in the velocity be so abrupt that the electronic wave function remained the same. What is the probability to find at $t>0$ the moving hydrogen atom with the electron in the ground state?

Solution:

As we learned the wave function of the electron at the new rest frame of the proton is

$$\tilde{\psi}(\vec{x}',t)=\exp\left(\frac{-i}{\hbar}\left[m\vec{v}\vec{x}-\frac{mv^2}{2}t\right]\right)\psi_{1,0}(\vec{x},t)=\frac{1}{\sqrt{\pi a_B^3}}\exp\left[\frac{-r}{a_B}-\frac{i}{\hbar}\left(mvz-\frac{mv^2}{2}t\right)\right]$$

Therefore at *t=0+* the electron wave function would be

$$\psi(\vec{x}')=\frac{1}{\sqrt{\pi a_B^3}}\exp\left[\frac{-r}{a_B}-\frac{i}{\hbar}mvz\right]$$

The ground state of the electron in the moving atom is described by the wave function

$$\psi_{1,0}(\vec{x}')\equiv\frac{1}{\sqrt{\pi a_B^3}}\exp\left(\frac{-r'}{a_B}\right)$$

Note that $\vec{x}=\vec{x}'$ at *t=0*. The probability P that the electron remains in the ground state is

$$P=\left|\int\psi(\vec{x})\psi_{1,0}(\vec{x})d\vec{x}'\right|^2=\frac{1}{\left(\pi a_B^3\right)^2}\left|\int\exp\left[-\frac{2r}{a_B}-\frac{i}{\hbar}mvz\right]d\vec{x}'\right|^2$$

Now we can use polar coordinates: momentum $z=r\cos\theta$; $\quad d\vec{x}=r^2drd(\cos\theta)d\varphi$

$$P=\frac{1}{\left(\pi a_B^3\right)^2}\left|\int\exp\left[-r\left(\frac{2}{a_B}-\frac{i}{\hbar}mv\cos\theta\right)\right]r^2drd(\cos\theta)d\varphi\right|^2$$

Integrals over φ and over θ can be evaluated straightforwardly. The result is

$$P=\frac{4\hbar^2}{a_B^6m^2v^2}\left(\operatorname{Im}\int_0^\infty\exp\left[-r\left(\frac{2}{a_B}-\frac{imvr}{\hbar}\right)\right]rdr\right)^2=\frac{4\hbar^2}{a_B^6m^2v^2}\left(\operatorname{Im}\left[\frac{1}{\left(2/a_B-imvr/\hbar\right)^2}\right]\right)^2$$

Integral $\int\exp(-cr)rdr$ can be evaluated by parts: $\int_0^\infty\exp(-cr)rdr=1/c^2$. Therefore

$$P=\frac{4\hbar^2}{a_B^6m^2v^2}\left(\operatorname{Im}\int_0^\infty\exp\left[-r\left(\frac{2}{a_B}-\frac{imvr}{\hbar}\right)\right]rdr\right)^2=\frac{4\hbar^2}{a_B^6m^2v^2}\left(\operatorname{Im}\left[\frac{1}{\left(2/a_B-imvr/\hbar\right)^2}\right]\right)^2$$

c) What is the probability to find the electron in the state with $n=2,l=1,m=1$?

Solution

This probability vanishes after the integration over the polar angle because $\psi_{2,1,1}\propto e^{i\varphi}$

Gyulassy

Sec. 3 QM

2

3 QM Quals 2010:

a) [10] Prove that the expectation value of the Hamiltonian $E[\phi]$ is stationary in the neighborhood of a discrete eigenstate, i.e., if $H\psi_n = E_n\psi_n$ and $\psi = \psi_n + \delta\psi$, then $\delta\langle\psi|H|\psi\rangle = 0$. Show also that $E[\phi] \geq E_0$ where $E_0 \leq E_n$ is the ground state energy.

b) [10] Apply the above to estimate the quantum ground state energy of a simple harmonic oscillator using a trial wavefunction of the form $\psi(x) = \exp\left(-x^2/a^2\right)$. Determine a, and compare $E[\psi]$ to the exact E_0 ground state energy. (Useful integrals are $\int_{-\infty}^{\infty} dx e^{-b^2x^2} = \sqrt{\pi}/b$ and its derivative with respect to b)

Quals2010 QM sec3 prob2a

QM MG 1

a) $E[\psi] = \frac{\langle\psi|H|\psi\rangle}{\langle\psi|\psi\rangle}$ mean energy in state $|\psi\rangle$

For ground state $H|\psi_0\rangle = E_0|\psi_0\rangle$, $E[\psi_0] = E_0$

Try variation $\psi = \psi_0 + \delta\psi$

$$E[\psi_0+\delta\psi] - E[\psi_0] = \frac{\langle\delta\psi|H|\psi_0\rangle + \langle\psi_0|H|\delta\psi\rangle}{\langle\psi_0|\psi_0\rangle} - \langle\psi_0|H|\psi_0\rangle\left(\frac{\langle\delta\psi|\psi_0\rangle + \langle\psi_0|\delta\psi\rangle}{\langle\psi_0|\psi_0\rangle^2}\right)$$

to first order

Let us normalize ground $\langle\psi_0|\psi_0\rangle = 1$

$$= E_0\left(\langle\delta\psi|\psi_0\rangle + \langle\psi_0|\delta\psi\rangle\right) - E_0\left(\quad '' \quad '' \quad \right)$$

$$= 0$$

Thus $\underline{E[\psi_0+\delta\psi] = E[\psi_0]} + \underset{\text{to first order}}{\underline{\underline{0}}} + \left(\begin{matrix}2^{nd}\text{ order}\\ \text{in } \delta\psi\end{matrix}\right)$

To show that $E[\psi] \geq E[\psi_0]$

$|\psi\rangle = |\psi+\delta\psi\rangle = \sum_n z_n|\psi_n\rangle$ we can expand in complete orthonormal

where $H|\psi_n\rangle = E_n|\psi_n\rangle$, $E_n \geq E_0$, $\langle\psi_n|\psi_m\rangle = \delta_{nm}$

$$\langle\psi|H|\psi\rangle = \sum_n\sum_m z_n^* z_m E_m \langle\psi_n|\psi_m\rangle$$

$$= \sum_n |z_n|^2 E_n \geq \left(\sum_n |z_n^2|\right) E_0 = E_0\langle\psi|\psi\rangle$$

$$\Rightarrow \frac{\langle\psi|H|\psi\rangle}{\langle\psi|\psi\rangle} \geq E_0$$

Quals2010 QM sec3 prob2b

MG 2

b) $\psi = e^{-x^2/a^2}$, Norm $= \mathcal{N} = \int_{-\infty}^{\infty} |\psi|^2 dx = a\sqrt{\frac{\pi}{2}}$ (from hint)

$\hat{H} = \frac{-\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{1}{2}m\omega^2 x^2 = \hat{KE} + \hat{V}$

Use $\langle\psi|\hat{KE}|\psi\rangle = -\hbar^2/2m \int (-)\left(\frac{d\psi}{dx}\right)^2 dx$ (int. by parts)

$= \frac{\hbar^2}{2m}\int\left(\frac{2x}{a^2}\right)^2|\psi|^2 = \left(\frac{2\hbar^2}{ma^4}\right)\int x^2|\psi|^2 = \left(\,''\,\right)\underbrace{\left(\frac{-d}{d(2/a^2)}\langle\psi|\psi\rangle\right)}_{\langle\psi|x^2|\psi\rangle}$

$= \frac{2\hbar^2}{ma^4}\left(\frac{a^3}{4}\frac{d}{da}\right) a\sqrt{\pi/2} = \frac{\hbar^2}{2ma^2}\langle\psi|\psi\rangle$

$\Rightarrow \langle E\rangle = \frac{\langle\psi|H|\psi\rangle}{\langle\psi|\psi\rangle} = \frac{\hbar^2}{2ma^2} + \frac{1}{8}m\omega^2 a^2$

Minimize w.r.t. a^2 $\quad \frac{d\langle E\rangle}{da^2} = 0 = -\frac{\hbar^2}{2ma^4} + \frac{1}{8}m\omega^2$

$\Rightarrow$ $a^2 = \sqrt{\frac{4\hbar^2}{m^2\omega^2}} = \frac{2\hbar}{m\omega}$ $\qquad \Rightarrow \langle KE\rangle = \langle V\rangle$

$\langle E\rangle_{min} = \frac{\hbar^2}{2m}\left(\frac{m\omega}{2\hbar}\right) + \frac{1}{8}m\omega^2\frac{2\hbar}{m\omega} = \frac{1}{2}\hbar\omega$

This is the exact quantum oscillator ground state energy

By part (a) if we tried any other guess for ψ we would obtain a larger energy

Quals2010 QM sec3 prob2a

QM

a) $E[\psi] = \frac{\langle\psi|H|\psi\rangle}{\langle\psi|\psi\rangle}$ mean energy in state $|\psi\rangle$

For ground state $H|\psi_0\rangle = E_0|\psi_0\rangle$, $E[\psi_0] = E_0$

Try variation $\psi = \psi_0 + \delta\psi$

$$E[\psi_0+\delta\psi] - E[\psi_0] = \frac{\langle\delta\psi|H|\psi_0\rangle + \langle\psi_0|H|\delta\psi\rangle}{\langle\psi_0|\psi_0\rangle} - \langle\psi_0|H|\psi_0\rangle\left(\frac{\langle\delta\psi|\psi_0\rangle + \langle\psi_0|\delta\psi\rangle}{\langle\psi_0|\psi_0\rangle^2}\right)$$

to first order

let us normalize ground $\langle\psi_0|\psi_0\rangle = 1$

$$= E_0\left(\langle\delta\psi|\psi_0\rangle + \langle\psi_0|\delta\psi\rangle\right) - E_0\left(\quad " \quad " \quad \right)$$

$$= 0$$

Thus $\underline{E[\psi_0+\delta\psi] = E[\psi_0] + \underline{\underline{0}} + \left(\text{2nd order in } \delta\psi\right)}$ to first order

To show that $E[\psi] \geq E[\psi_0]$

we can expand in complete orthonormal

$$|\psi\rangle = |\psi+\delta\psi\rangle = \sum_n z_n |\psi_n\rangle$$

where $H|\psi_n\rangle = E_n|\psi_n\rangle$, $E_n \geq E_0$, $\langle\psi_n|\psi_m\rangle = \delta_{nm}$

$$\langle\psi|H|\psi\rangle = \sum_n\sum_m z_n^* z_m E_m \langle\psi_n|\psi_m\rangle$$

$$= \sum_n |z_n|^2 E_n \geq \left(\sum_n |z_n^2|\right) E_0 = E_0\langle\psi|\psi\rangle$$

$$\Rightarrow \frac{\langle\psi|H|\psi\rangle}{\langle\psi|\psi\rangle} \geq E_0$$

Quals2010 QM sec3 prob2b

MG 2

b) $\psi = e^{-x^2/a^2}$, Norm $= \mathcal{N} = \int_{-\infty}^{\infty} |\psi|^2 dx = a\sqrt{\frac{\pi}{2}}$ (from hint)

$$\hat{H} = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{1}{2}m\omega^2x^2 = \hat{KE} + \hat{V}$$

Use $\langle\psi|\hat{KE}|\psi\rangle = -\hbar^2/2m \int (-)\left(\frac{d\psi}{dx}\right)^2 dx$ (int. by parts)

$$= \frac{\hbar^2}{2m}\int\left(\frac{2x}{a^2}\right)^2|\psi|^2 = \left(\frac{2\hbar^2}{ma^4}\right)\int x^2|\psi|^2 = \left(\,''\,\right)\underbrace{\left(\frac{-d}{d(2/a^2)}\langle\psi|\psi\rangle\right)}_{\langle\psi|x^2|\psi\rangle}$$

$$= \frac{2\hbar^2}{ma^4}\left(\frac{a^3}{4}\frac{d}{da}\right) a\sqrt{\pi/2} = \frac{\hbar^2}{2ma^2}\langle\psi|\psi\rangle$$

$$\Rightarrow \langle E\rangle = \frac{\langle\psi|H|\psi\rangle}{\langle\psi|\psi\rangle} = \frac{\hbar^2}{2ma^2} + \frac{1}{8}m\omega^2a^2$$

Minimize w.r.t. a^2 $\quad \frac{d\langle E\rangle}{da^2} = 0 = -\frac{\hbar^2}{2ma^4} + \frac{1}{8}m\omega^2$

$$\Rightarrow \underline{a^2 = \sqrt{\frac{4\hbar^2}{m^2\omega^2}} = \frac{2\hbar}{m\omega}} \qquad \Rightarrow \underline{\langle KE\rangle = \langle V\rangle}$$

$$\langle E\rangle_{min} = \frac{\hbar^2}{2m}\left(\frac{m\omega}{2\hbar}\right) + \frac{1}{8}m\omega^2\frac{2\hbar}{m\omega} = \frac{1}{2}\hbar\omega$$

This is the exact quantum oscillator ground state energy ☐

By part (a) if we tried any other guess for ψ we would obtain a larger energy ☐

GYULASSY
Section 3 QM
Problem 2,
Parts a and b

Quals2010 QM sec3 prob2a

QM MG 1

a) $E[\psi] = \frac{\langle\psi|H|\psi\rangle}{\langle\psi|\psi\rangle}$ mean energy in state $|\psi\rangle$

For ground state $H|\psi_0\rangle = E_0|\psi_0\rangle$, $E[\psi_0] = E_0$

Try variation $\psi = \psi_0 + \delta\psi$

$$E[\psi_0+\delta\psi] - E[\psi_0] = \frac{\langle\delta\psi|H|\psi_0\rangle + \langle\psi_0|H|\delta\psi\rangle}{\langle\psi_0|\psi_0\rangle}$$

to first order

$$- \langle\psi_0|H|\psi_0\rangle \left(\frac{\langle\delta\psi|\psi_0\rangle + \langle\psi_0|\delta\psi\rangle}{\langle\psi_0|\psi_0\rangle^2}\right)$$

Let us normalize ground $\langle\psi_0|\psi_0\rangle = 1$

$$= E_0(\langle\delta\psi|\psi_0\rangle + \langle\psi_0|\delta\psi\rangle)$$
$$- E_0(\quad " \quad " \quad)$$
$$= 0$$

Thus $E[\psi_0+\delta\psi] = E[\psi_0] + \underline{\underline{0}} + \left(\begin{matrix}\text{2nd order}\\ \text{in } \delta\psi\end{matrix}\right)$
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To show that $E[\psi] \geq E[\psi_0]$

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$|\psi\rangle = |\psi+\delta\psi\rangle = \sum_n z_n |\psi_n\rangle$

where $H|\psi_n\rangle = E_n|\psi_n\rangle$, $E_n \geq E_0$, $\langle\psi_n|\psi_m\rangle = \delta_{nm}$

$$\langle\psi|H|\psi\rangle = \sum_n\sum_m z_n^* z_m E_m \langle\psi_n|\psi_m\rangle$$
$$= \sum_n |z_n|^2 E_n \geq \left(\sum_n |z_n^2|\right) E_0 = E_0\langle\psi|\psi\rangle$$

$$\Rightarrow \frac{\langle\psi|H|\psi\rangle}{\langle\psi|\psi\rangle} \geq E_0$$

Quals2010 QM sec3 prob2b

MG 2

b) $\psi = e^{-x^2/a^2}$, Norm $= \mathcal{N} = \int_{-\infty}^{\infty} |\psi|^2 dx = a\sqrt{\frac{\pi}{2}}$ (from hint)

$\hat{H} = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{1}{2}m\omega^2 x^2 = \hat{KE} + \hat{V}$

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$\Rightarrow \langle E\rangle = \frac{\langle\psi|H|\psi\rangle}{\langle\psi|\psi\rangle} = \frac{\hbar^2}{2ma^2} + \frac{1}{8}m\omega^2 a^2$

Minimize w.r.t. a^2 $\quad \frac{d\langle E\rangle}{da^2} = 0 = -\frac{\hbar^2}{2ma^4} + \frac{1}{8}m\omega^2$

$\Rightarrow \underline{a^2 = \sqrt{\frac{4\hbar^2}{m^2\omega^2}} = \frac{2\hbar}{m\omega}}$ $\quad \Rightarrow \underline{\langle KE\rangle = \langle V\rangle}$

$\langle E\rangle_{min} = \frac{\hbar^2}{2m}\left(\frac{m\omega}{2\hbar}\right) + \frac{1}{8}m\omega^2\frac{2\hbar}{m\omega} = \frac{1}{2}\hbar\omega$

This is the exact quantum oscillator ground state energy

By part (a) if we tried any other guess for ψ we would obtain a larger energy

QM Kim

Sec. 3 QM

#3

2009 Qual Problems – Philip Kim

QM

Electron confined in a ring with magnetic field.

Consider an electron of charge e and mass m confined in a ring of radius R. In a cylindrical coordinate the Hamiltonian of this confined system can be described by

$$H = \frac{1}{2m}\left(\frac{\hbar}{i}\nabla\right)^2 = -\frac{\hbar^2}{2m}\left(\frac{1}{R}\frac{d}{d\varphi}\right)^2.$$

where φ is the azimuthal angle.

(a) Find the energy eigen values and normalized eigen wavefunctions of this system.

(b) Now we consider a magnetic field $\vec{B} = B\hat{z}$ applied to z-direction. Employing the symmetry gauge, the corresponding vector potential on the ring can be expressed by

$$\vec{A} = \frac{BR}{2}\hat{\varphi},$$

where $\hat{\varphi}$ is the unit vector along the azimuthal angle φ. In the magnetic field, the Hamitonian is given by $H = \frac{1}{2m}\left(\frac{\hbar}{i}\nabla - e\vec{A}\right)^2$. Find the energy eigen value of a confined electron in this ring in the presence of a fixed magnetic field.

(c) Find the smallest magnetic field at which one can find the non-degenerate ground and doubly degenerate excited states?

There are some changes in solution.
New solution attached.

Corrected solution.

QM

Electron confined in a ring

(a) Ignoring spin,

$$H = -\frac{\hbar^2}{2m}\left(\frac{1}{R}\frac{d}{d\varphi}\right)^2$$

Try $\psi(\varphi) = A e^{-i\lambda\varphi} \Rightarrow E_\lambda = \frac{\hbar^2}{2m}\left(\frac{\lambda}{R}\right)^2$

Applying periodic boundary condition,

$$\psi(\varphi+2\pi) = \psi(\varphi) \Rightarrow \lambda = 0, \pm1, \pm2, \cdots$$

Normalization

$$1 = \int_0^{2\pi} d\varphi\, |\psi|^2 = 2\pi A^2 \Rightarrow A = \frac{1}{\sqrt{2\pi}}$$

$$\Rightarrow \psi_n(\varphi) = \frac{1}{\sqrt{2\pi}} e^{-in\varphi}, \quad E_n = \frac{\hbar^2}{2m}\left(\frac{n}{R}\right)^2, \quad n = 0, \pm1, \pm2,$$

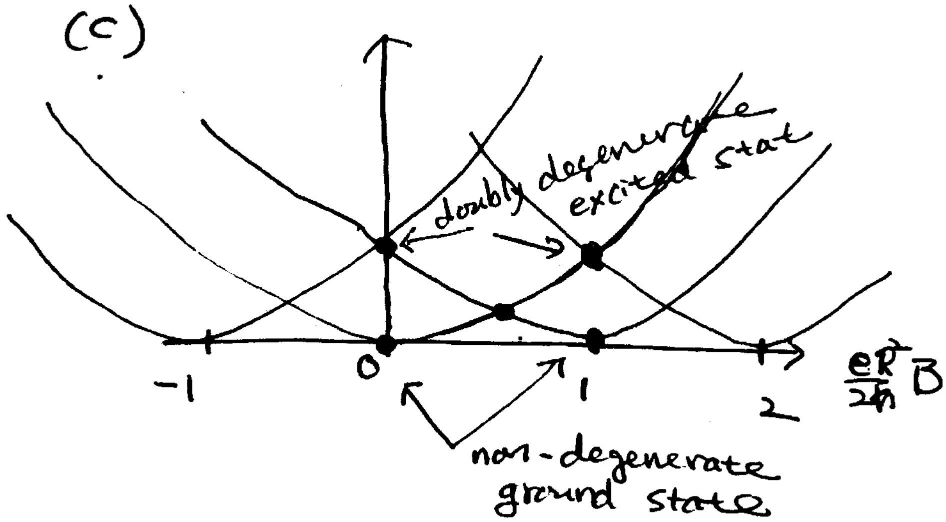
(b) $\vec{A} = \frac{BR}{2}\hat{\varphi}$

$$\tilde{H} = \frac{1}{2m}\left[\frac{\hbar}{i}\frac{1}{R}\frac{\partial}{\partial\varphi} - \frac{eBR}{2}\right]^2 = \frac{\hbar^2}{2mR^2}\left[\frac{1}{i}\frac{\partial}{\partial\varphi} - \frac{eBR^2}{2\hbar}\right]^2$$

$\psi_n(\varphi)$ in (a) becomes eigenfunction.

$$E_n = \frac{\hbar^2}{2mR^2}\left[n - \frac{eBR^2}{2\hbar}\right]^2$$

(c)



$B_{min} = 0$

(for the smallest).

or

$B_{min} = \frac{2\hbar}{eR^2}$

(for the smallest non-zero field)

Columbia Physics Department
2010 QUALIFYING EXAMS

All questions are to be scored on a scale of
<u>0</u> to <u>15</u>
(0 = failing, 15 = highest possible score)

Please write the numerical score in <u>red</u> ink directly on the cover of the exam booklet.

Please be sure to read the problem <u>as it appears in the exam.</u> Some problems have been edited. Make sure that you are grading what the students were asked.

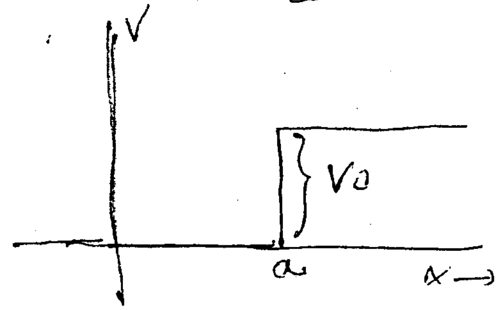
Please return the graded exam booklets to Lalla or to Rasma in 704 Pupin, ideally within 24 hours, or as soon as possible.

Thanks!

Quantum Mechanics

QM Mueller
Sec 3 QM
4

Consider a particle of mass m in a one-dimensional potential $V(x)$ where



$V(x) = \infty \qquad x<0$
$V(x) = 0 \qquad 0<x<a$
$V(x) = V_0 \qquad x>a$

with $V_0 > 0$.

(i) If $E = \frac{\hbar^2 k^2}{2m}$ is a bound state energy and $V_0 - E = \frac{\hbar^2 K^2}{2m}$, give the equation determining possible values of E.

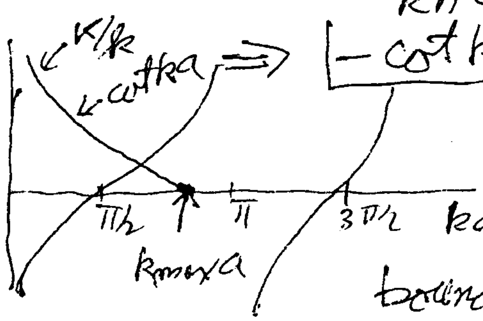
(ii) Give the condition on V_0 and a for at least one bound state to exist.

(iii) What are the energy levels when $V_0 = \infty$

Solution:

(i) $0<x<a \qquad u(x) = A \sin kx$
$a<x<\infty \qquad u(x) = B e^{-Kx}$
match at $x=a$: $\quad A \sin ka = B e^{-Ka}$
$\qquad kA \cos ka = -BK e^{-Ka}$

(ii)



$\Rightarrow \boxed{-\cot ka = K/k}$

$k_{max} = \sqrt{2mV_0}/\hbar$

$k_{max}\, a = \sqrt{2mV_0}\, a/\hbar$

bound state if $\boxed{\sqrt{2mV_0}\, a/\hbar > \pi/2}$

(iii) $V_0 = \infty \Rightarrow K = \infty \Rightarrow \cot ka = -\infty \Rightarrow ka = n\pi$

$\boxed{E_n = \left(\frac{n\pi\hbar}{a}\right)^2 \frac{1}{2m}} \qquad n = 1, 2, 3, \cdots$

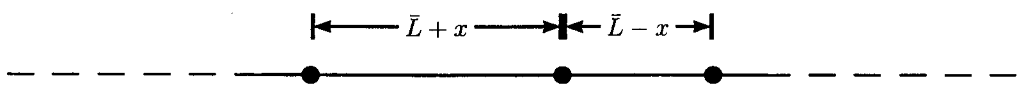
QM Ponton
Sec 3 QM
#5
(with corrected solution)

Qualifying exam 2010
Eduardo Pontón

1. **Quantum mechanics**

Consider a quantum system with an infinite set of particles in one dimension as shown in the figure. Particles cannot cross neighbors. We are interested in the probability distribution of spacings L between a particle and its neighbor to the left, given that the average spacing between particles is $\bar{L}$.

In the simplest approximation to this many-body problem, a single particle moves between two fixed neighbors separated by $2\bar{L}$. Let $x = L - \bar{L}$ denote the deviation from the midpoint.



(a) Find the probability distribution $P(L)$ in the ground state, in the above approximation, and assuming there are no interparticle interactions (other than contact interactions).

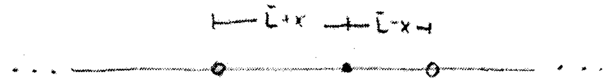
(b) Now suppose there are strong repulsive potentials between pairs of neighboring particles of the form AL^{-n}. Considering only the nearest neighbors, write the potential energy for the middle particle for $x \ll \bar{L}$. Write the result explicitly in terms of A, n and $\bar{L}$.

(c) Still assuming that $x \ll \bar{L}$ and that the neighboring particles are fixed, write the Schrödinger equation for the middle particle, and argue that the problem can be mapped into a familiar one. Based on this analogy, what is the form of the distribution $P(L)$? How does its *width* scale with $\bar{L}$?

(d) In the limit of strong repulsion, as in *b*) and *c*) above, explain how you can measure the power n and the amplitude A that characterize the potential.

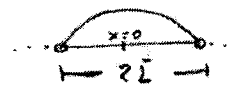
over →

Soln



a) Free particle with boundary conditions $\psi(-\bar{L}) = \psi(\bar{L}) = 0$

Ground state : $\psi_0(x) = N \sin \frac{\pi(\bar{L}+x)}{2\bar{L}}$



$$E_0 = \frac{\hbar^2}{2m} \cdot \left(\frac{\pi}{2\bar{L}}\right)^2 = \frac{\pi^2 \hbar^2}{2m\bar{L}^2}$$

$$N^{-2} = \int_{-\bar{L}}^{\bar{L}} dx \sin^2 \frac{\pi(\bar{L}-x)}{2\bar{L}} = \bar{L}$$

For spacing from particle to the left : $L = \bar{L} + x$

$$P(L) = |\psi_0(x)|^2 = \frac{1}{\bar{L}} \sin^2 \frac{\pi L}{2\bar{L}}$$

b)
$$V(x) = A(\bar{L}+x)^{-n} + A(\bar{L}-x)^{-n}$$
$$= \frac{2A}{\bar{L}^n} \left\{ 1 + \frac{1}{2} n(n+1) \left(\frac{x}{\bar{L}}\right)^2 + O(x^3/\bar{L}^3) \right\}$$
$$= \frac{2A}{\bar{L}^n} + Cx^2 + \ldots$$
$$C = \frac{n(n+1)A}{\bar{L}^{n+2}} \equiv \frac{1}{2} m\omega^2$$

c) Natural length scale (from m & ω) : $a = \left(\sqrt{\frac{m\omega}{\hbar}}\right)^{-1}$

Ground state for harmonic osc. $\propto e^{-\frac{x^2}{2a^2}}$ (Gaussian)

$\rightarrow$ width $\sim a \sim \omega^{-1/2} \sim \left(\frac{1}{\bar{L}^{\frac{n}{4}+\frac{1}{2}}}\right)^{-1}$

d) Measure width of $P(L)$ for different $\bar{L}$ (densities) to determine C as a function of $\bar{L}$ and fit for A and n.