Columbia University
Department of Physics
QUALIFYING EXAMINATION
Monday, January 11, 2010
3:10 PM - 5:10 PM

Electromagnetism
Section 2.

Two hours are permitted for the completion of this section of the examination. Choose **4 problems** out of the 5 included in this section. Remember to hand in **only** the 4 problems of your choice (if by mistake you hand in 5 problems, the highest scoring problem grade will be dropped). Apportion your time carefully.

Use separate answer booklet(s) for each question. Clearly mark on the answer booklet(s) which question you are answering (e.g., Section 2 (Electromagnetism), Question 2; Section 2 (Electromagnetism), Question 6; etc.)

Do **NOT** write your name on your answer booklets. Instead clearly indicate your **Exam Letter Code**.

You may refer to the single handwritten note sheet on 8 $1/2 \times 11$" paper (double-sided) you have prepared on Electromagnetism. The note sheet cannot leave the exam room once the exam has begun. This note sheet must be handed in at the end of today's exam. Please include your Exam Letter Code on your note sheet. No other extraneous papers or books are premitted.

Simple calculators are permitted. However, the use of calculators for storing and/or recovering formulae or constants is NOT permitted.

Questions should be directed to the proctor.

Good luck!!

1. Consider a rigid, ideally conducting sphere of radius R, with total charge equal to zero. The sphere rotates with angular velocity $\vec{\Omega}$; $\Omega R \ll c$. Suppose a dipole magnetic field threads the sphere. The dipole is centered on the center of the sphere. The dipole moment $\vec{\mu}$ is given; it is aligned with $\vec{\Omega}$.

   (a) What voltage is induced between the equator and the poles of the sphere?

   (b) Find the charge density $\rho(r, \theta)$ established inside the sphere. Here r and θ are spherical coordinates: r is the distance from the center and θ is the polar angle measured from the rotational axis.

   (c) Find the electric field outside the sphere.

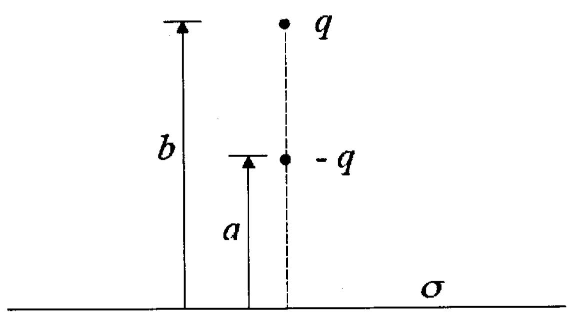
   Hint: any axisymmetric solution of $\nabla^2\Phi = 0$ that vanishes at infinity has the following form in spherical coordinates r, θ, ϕ

$$\Phi(r,\theta) = \sum_{n=0}^{\infty} a_n \left(\frac{r}{R}\right)^{-n-1} P_n(\cos\theta), \qquad [P_0 = 1,\ P_1 = \cos\theta,\ P_2 = \frac{3\cos^2\theta - 1}{2},\ ...]$$

2. A positive point charge q is fixed 1 cm above a horizontal, grounded conducting x-y plane. An equal negative charge $-q$ can be moved along the perpendicular dropped from q to the plane.

   (a) Where should $-q$ be placed for the total force on it to be zero?

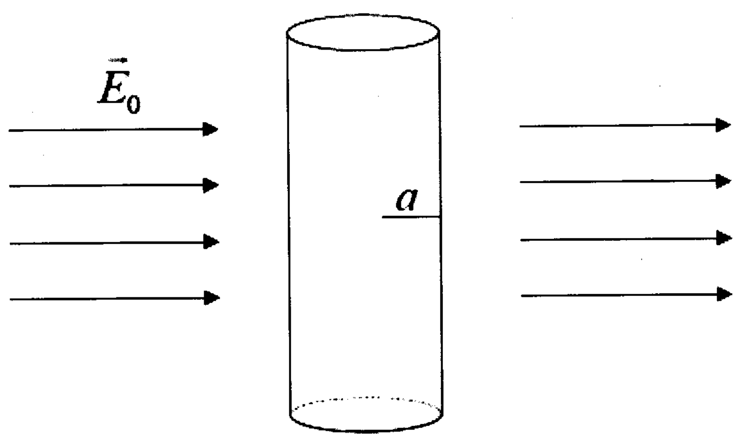
   (b) Taking the distance between q and the plane equal to b, and the distance from $-q$ to the plane equal to be a, what is the surface charge density, $\sigma(x, y)$, on the conductor? Express your answer in terms of a, b, q, x and y.



3. Consider an infinitely long, grounded conducting cylinder, of radius a, which is introduced into a uniform electric field $\vec{E}_0$. The axis of the cylinder is perpendicular to $\vec{E}_0$.

   (a) Find an expression for the external potential after insertion of the cylinder.

   (b) Find an expression for the surface charge induced on the cylinder.



4. A perpendicularly incident beam of right circularly polarized light is reflected by an ideal stationary mirror. Show that the reflected beam is left circularly polarized.

5. A magnetic monopole is a hypothetical particle that is a source for a Coulomb magnetic field

$$\vec{\mathbf{B}} = \frac{g\hat{\mathbf{r}}}{r^2}$$

(a) Consider a particle with mass m and electric charge q that is moving in the magnetic field of a static magnetic monopole. Show that the usual expression for angular momentum, $\vec{\mathbf{r}} \times (m\vec{\mathbf{v}})$ is not conserved, but that there is a conserved angular momentum of the form

$$\vec{\mathbf{L}} = \vec{\mathbf{r}} \times (m\vec{\mathbf{v}}) + \vec{\mathbf{f}}.$$

Determine $\vec{\mathbf{f}}$.

(b) An electric charge q and a magnetic charge g with fixed positions are located a distance D apart. The combined fields of these charges have a nonzero angular momentum.

i. Show that the magnitude of this angular momentum does not depend on the distance D.

ii. Determine the magnitude and direction of the angular momentum.

The integral

$$\int_0^\infty dy \frac{y}{(y^2 - 2ay + 1)^{3/2}} = \frac{1}{1-a}$$

may be useful.

E+M Beloborodov
Sec. 2 E+M
#1

E&M:

Consider a rigid, ideally conducting sphere of radius R with total charge equal to zero. The sphere rotates with angular velocity $\vec{\Omega}$; $\Omega R \ll c$. Suppose a dipole magnetic field threads the sphere. The dipole is centered on the center of the sphere. The dipole moment $\vec{\mu}$ is given; it is aligned with $\vec{\Omega}$.

(a) What voltage is induced between the equator and the poles of the sphere?

(b) Find the charge density $\rho(r,\theta)$ established inside the sphere. Here r and θ are spherical coordinates: r is the distance from the center and θ is the polar angle measured from the rotational axis.

(c) Find the electric field outside the sphere. Hint: any axisymmetric solution of $\nabla^2\Phi = 0$ that vanishes at infinity has the following form in spherical coordinates r, θ, ϕ

$$\Phi(r,\theta) = \sum_{n=0}^{\infty} a_n \left(\frac{r}{R}\right)^{-n-1} P_n(\cos\theta), \qquad [P_0 = 1,\ P_1 = \cos\theta,\ P_2 = \frac{3\cos^2\theta - 1}{2},\ ...] \tag{1}$$

Solution:

(a) Electric field in the frame co-rotating with the sphere vanishes inside the ideal conductor: $\vec{E}' = 0$. In the static lab frame, electric field is induced by rotation $\vec{v}_{\rm rot} = \vec{\Omega} \times \vec{r}$:

$$\vec{E} = \vec{E}' - \frac{\vec{v}_{\rm rot} \times \vec{B}}{c} = -\frac{\Omega r \sin\theta}{c}\, \vec{e}_\phi \times \vec{B}, \tag{2}$$

where $\vec{e}_\phi$ is the unit vector in the ϕ-direction of the spherical coordinate system r, θ, ϕ. The dipole magnetic field is given by

$$\vec{B} = \vec{B}' = \frac{3(\vec{\mu}\cdot\vec{e}_r)\vec{e}_r - \vec{\mu}}{r^3} = \frac{\mu}{r^3}\left(2\cos\theta\, \vec{e}_r + \sin\theta\, \vec{e}_\theta\right), \tag{3}$$

where $\vec{e}_r$ and $\vec{e}_\theta$ are the unit vectors in the r and θ directions. Substitution of (2) to (1) gives

$$\vec{E} = \frac{\mu\Omega}{cr^2}\sin\theta\left(\sin\theta\, \vec{e}_r - 2\cos\theta\, \vec{e}_\theta\right), \qquad r < R.$$

Since $\nabla \times \vec{E} = -c^{-1}\partial\vec{B}/\partial t = 0$, the electric field is potential, $\vec{E} = -\nabla\Phi$.

$$\Phi(R,\theta) - \Phi(R,0) = -\int_0^\theta E_\theta\, R\, d\theta = \frac{\mu\Omega}{cR^2}\int_0^\theta \sin 2\theta\, d\theta = \frac{\mu\Omega}{cR^2}\sin^2\theta. \tag{4}$$

The potential difference between the equator and the poles is $\Phi(R,\pi/2) - \Phi(R,0) = \mu\Omega/cR^2$.

(b)

$$\rho = \frac{\nabla\cdot\vec{E}}{4\pi} = \frac{1}{4\pi r^2\sin\theta}\left[\partial_r(r^2\sin\theta E_r) + \partial_\theta(r\sin\theta E_\theta)\right] = -\frac{\mu\Omega}{2\pi c r^3}\left(2\cos^2\theta - \sin^2\theta\right).$$

(c) Potential Φ satisfies Laplace equation $\nabla^2\Phi = 0$ at $r > R$ and has the form (1). The boundary condition for Φ at $r = R$ is given by eq. (4). Since $\sin^2\theta = (2/3)(P_0 - P_2)$, the boundary condition expanded in Legendre polynomials reads

$$\Phi(R,\theta) = \left[\Phi(R,0) + \frac{2\mu\Omega}{3cR}\right]P_0 - \frac{2\mu\Omega}{3cR}P_2 \qquad \Rightarrow \quad a_0 = \Phi(R,0) + \frac{2\mu\Omega}{3cR}, \quad a_2 = -\frac{2\mu\Omega}{3cR}.$$

The boundary condition selects the two non-zero a_n ($n = 0, 2$) in eq. (1). However, since the total charge of the sphere is zero, the monopole contribution must vanish, $a_0 = 0$ [it implies $\Phi(R,0) = -2\mu\Omega/3cR$]. Thus, one finds at $r > R$

$$\Phi(r,\theta) = -\frac{2\mu\Omega R^2}{3cr^3}P_2(\cos\theta), \qquad E_r = -\frac{\partial\Phi}{\partial r} = \frac{\mu\Omega R^2}{cr^4}(3\cos\theta - 1), \qquad E_\theta = -\frac{1}{r}\frac{\partial\Phi}{\partial\theta} = -\frac{\mu\Omega R^2}{cr^4}\sin 2\theta.$$

EM Brodsjman

Sec 2 E+M

2

Quals 10, EM

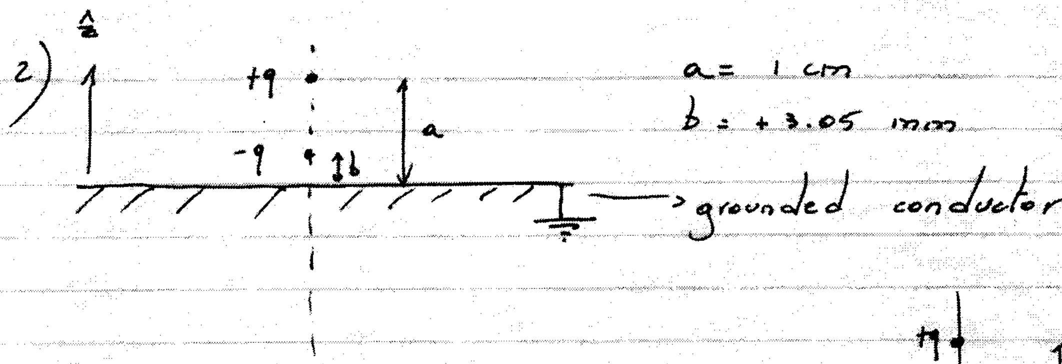
December 2, 2009

Problem

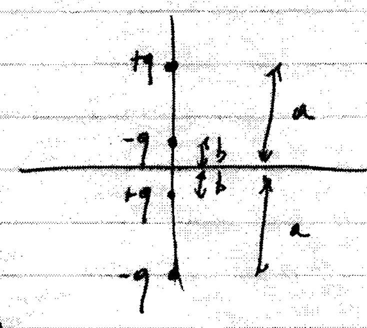
A positive point charge q is fixed 1 cm above a horizontal, grounded conducting plane. An equal negative charge $-q$ can be moved along the perpendicular dropped from q to the plane. Where should $-q$ be placed for the total force on it to be zero? Taking the distance between q and the plane equal to b, and the distance from $-q$ to the plane equal to a, what is the surface charge density on the conductor?

2) $a = 1$ cm

$b = +3.05$ mm



This is an image problem: take



The force on the charge at b is then

$$F_z = \frac{q^2}{4\pi\varepsilon_0}\left[\underbrace{\frac{1}{(a-b)^2}}_{\text{due to } +q \text{ at } b} - \underbrace{\frac{1}{(2b)^2}}_{\text{due to } +q \text{ at } -b} + \underbrace{\frac{1}{(a+b)^2}}_{\text{due to } -q \text{ at } -a}\right]$$

$$F_z = 0 \quad \text{if} \quad \frac{1}{(a-b)^2} - \frac{1}{(2b)^2} + \frac{1}{(a+b)^2} = 0$$

Both a and b are > 0, and we will assume $a \neq b$

$$\Rightarrow (a+b)^2(2b)^2 - (a-b)^2(a+b)^2 + (a-b)^2(2b)^2 = 0$$

$$\Leftrightarrow 4a^2b^2 + 4ab^3 + 4b^4 - a^4 + 2a^3b - a^2b^2 - 2a^3b + 4a^2b^2 - 2ab^3 - a^2b^2 + 2ab^3 - b^4 = 0$$

$$\Leftrightarrow 10a^2b^2 + 7b^4 - a^4 = 0$$

$$\Leftrightarrow 7b^4 + 10a^2b^2 - a^4 = 0$$

$$\underline{\text{Sol.}} \quad x = \frac{-10a^2 \pm \sqrt{100a^4 + 28a^4}}{14}$$

$$a = 1 \quad \Rightarrow \quad x = \frac{-10 \pm \sqrt{128}}{14} = \frac{-10 \pm 11.3}{14}$$

The charge $-q$ is above the plane, (and b is a real number) so we take the positive solution:

$$x = \frac{1.3}{14} \Rightarrow b = \sqrt{\frac{1.3}{14}} = 0.305$$

$\Rightarrow b = 3.05$ mm implies the force on $-q = 0$

What is the surface charge density on the conductor?

- For a conductor, $\vec{E} = \frac{\sigma}{\epsilon_0}\hat{n}$ at the surface

$$\rightsquigarrow \sigma = \epsilon_0 E_{\hat{n}} = -\epsilon_0 \frac{\partial V}{\partial n}\bigg|_{surface}$$

$$\rightarrow \text{in this case, } \sigma = -\epsilon_0 \frac{\partial V}{\partial z}\bigg|_{z=0}$$

- $V = \frac{1}{4\pi\epsilon_0} \sum_i \frac{q_i}{|\Delta r_i|}$

$$= \frac{1}{4\pi\epsilon_0}\left[\frac{q}{\sqrt{x^2+y^2+(z-b)^2}} + \frac{-q}{\sqrt{x^2+y^2+(z-a)^2}} + \frac{q}{\sqrt{x^2+y^2+(z+a)^2}} + \frac{-q}{\sqrt{x^2+y^2+(z+b)^2}}\right]$$

$$\rightsquigarrow \frac{\partial V}{\partial z} = -\frac{1}{2}\frac{2q}{4\pi\epsilon_0}\left[\frac{(z-b)}{\left(x^2+y^2+(z-b)^2\right)^{3/2}} - \frac{(z-a)}{\left(x^2+y^2+(z-a)^2\right)^{3/2}} + \frac{(z+a)}{\left(x^2+y^2+(z+a)^2\right)^{3/2}} - \frac{(z+b)}{\left(x^2+y^2+(z+b)^2\right)^{3/2}}\right]$$

$$\Rightarrow \sigma = -\epsilon_0 \frac{\partial V}{\partial z}\bigg|_{z=0} = \frac{q}{4\pi}\left[\frac{2a}{(x^2+y^2+a^2)^{3/2}} - \frac{2b}{(x^2+y^2+b^2)^{3/2}}\right]$$

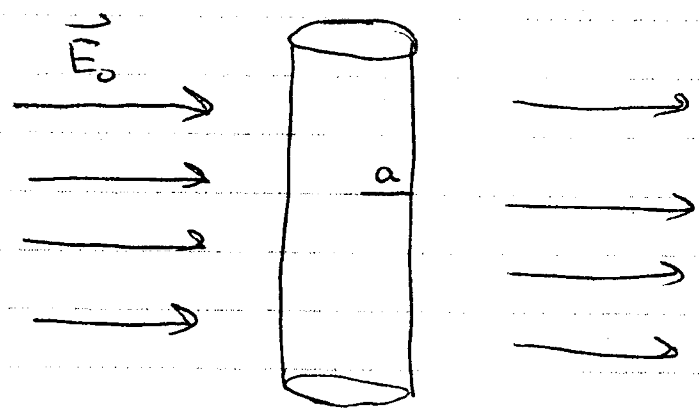
EM Hailey
Sec. 2 E+M
3

Problem 2: E-M = Hailey (Alternate)

Consider an infinitely long, grounded conducting cylinder which is introduced into a uniform electric field $\vec{E}_0$. The axis of the cylinder is perpendicular to $\vec{E}_0$.

a) Find an expression for the external potential after insertion of the cylinder.

b) Find an expression for the surface charge induced on the cylinder



E-M : Hurley

Solution 2 : This is just the 2-d Laplace equation in cylindrical coordinates for an infinitely long cylinder since there is no z-dependence.

$$\nabla^2\phi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} = 0$$

Separating $\phi(r\,\theta) = R(r)\,Q(\theta)$ $\Rightarrow$

$$\frac{r}{R}\frac{d}{dr}\left(r\frac{dR}{dr}\right) + \frac{1}{Q}\frac{d^2Q}{d\theta^2} = 0$$

Let the separation constant be n^2

$$d^2Q/d\theta^2 + n^2Q = 0\,; \quad \frac{r}{R}\frac{d}{dr}\left(r\frac{dR}{dr}\right) = n^2$$

$$Q(\theta) \sim e^{\pm in\theta}\,; \quad r\frac{d}{dr}\left(r\frac{dR}{dr}\right) - n^2R = 0$$

By simple substitution the radial equation has solutions $R(r) \sim r^{\pm n}$ so

$$\phi(\rho\theta) \sim \sum_n \left(a_n r^n + b_n r^{-n}\right) e^{\pm in\theta}$$

since $a_n = 0$ to prevent ϕ blowing up as $r \to \infty$ and noting the form of the uniform field

$$\phi_{ext} = -E_0\, r\cos\theta + \sum_n b_n r^{-n}\, e^{\pm in\theta}$$

$$\phi_{ext} = -E_0\, r\cos\theta + \frac{b_1}{r}\cos\theta$$

Only the $n=1$ term survives so that we can match the boundary condition $\phi(a\ \theta) = 0$

$$\phi_{ext}(a\,\theta) = 0 = -E_0 a \cos\theta + \frac{b_1}{a}\cos\theta$$

$$b_1 = E_0 a^2$$

$$\phi_{ext} = -E_0 r\cos\theta + \frac{E_0 a^2}{r}\cos\theta$$

a.) $\phi_{ext} = -E_0 r\cos\theta\left(1 - a^2/r^2\right)$ Ans

b.) The induced charge is just the normal component of the E-field ie $E_n = 4\pi\sigma$ from Gauss' Law

$$\sigma = 1/4\pi \ -\frac{\partial\phi_{ext}}{\partial r}\Big|_{r=a}$$

$$\sigma = \frac{E_0\cos\theta}{4\pi}\left(1 + \frac{a^2}{r^2}\right)\Big|_{r=a}$$

$$\sigma = \frac{E_0\cos\theta}{2\pi}$$ Ans

Marka
Sec. 2 E+M
4

#3: A perpendicularly incident beam of right circularly polarized light is reflected by a stationary mirror. Show that the reflected beam is left circularly polarized.

FROM THE FRESNEL EQUATIONS:

$$\vec{E}_R = \frac{1-\sqrt{\varepsilon}}{1+\sqrt{\varepsilon}}\,\vec{E}_I \underset{\varepsilon\to\infty}{\approx} -\vec{E}_I$$

RIGHT CIRCULAR POLARIZED:

$$E_y = A\cos(\omega t - kx)$$

$$E_z = A\sin(\omega t - kx)$$

LEFT CIRCULAR POLARIZED

$$E_y = A\cos(\omega t - kx)$$

$$E_z = -A\sin(\omega t - kx)$$

WHILE THE REFLECTION INTRODUCES THE SAME PHASE SHIFT FOR y & z components (RELATIVE PHASE SHIFT DOES NOT CHANGE), THE DIRECTION OF PROPAGATION IS REVERSED. THEREFORE THE "TWIST" OF THE POLARIZATION RELATIVELY TO THE DIRECTION OF PROPAGATION FLIPS.

Weinberg
Sec 2 E+M
#5

E&M problem

A magnetic monopole is a hypothetical particle that is a source for a Coulomb magnetic field

$$\mathbf{B} = \frac{g\hat{\mathbf{r}}}{r^2} \tag{1}$$

a) Consider a particle with mass m and electric charge q that is moving in the magnetic field of a static magnetic monopole. Show that the usual expression for angular momentum, $\mathbf{r} \times (m\mathbf{v})$ is not conserved, but that there is a conserved angular momentum of the form

$$\mathbf{L} = \mathbf{r} \times (m\mathbf{v}) + \mathbf{f} \tag{2}$$

Determine $\mathbf{f}$.

b) An electric charge q and a magnetic charge g with fixed positions are located a distance D apart. The combined fields of these charges have a nonzero angular momentum. (i) Show that the magnitude of this angular momentum does not depend on the distance D. (ii) Determine the magnitude and direction of the angular momentum. The integral

$$\int_0^\infty dy \, \frac{y}{(y^2 - 2ay + 1)^{3/2}} = \frac{1}{1-a} \tag{3}$$

may be useful.

a) $\frac{d}{dt}[\vec{r} \times (m\vec{v})] = \vec{v} \times (m\vec{v}) + \vec{r} \times (m\dot{\vec{v}})$

$= 0 + \vec{r} \times (q\vec{v} \times \vec{B})$

$= \vec{r} \times (q\vec{v} \times \frac{\hat{r}}{r^2})\, g$

$= qg\,\frac{1}{r}\,\hat{r} \times (\vec{v} \times \hat{r})$

$= qg\,\frac{1}{r}\,(\vec{v} - \hat{r}\cdot\vec{v}\,\hat{r})$

$= qg\,\frac{d}{dt}(\hat{r})$

$\Rightarrow L - qg\hat{r}$ is conserved

b) Use the fact that the linear momentum density is

$$\vec{p} = \frac{1}{c^2}\vec{S} = \frac{1}{c^2}\frac{1}{\mu_0}\vec{E}\times\vec{B} = \epsilon_0 \vec{E}\times\vec{B}$$

$$\Rightarrow \vec{L} = \epsilon_0 \int d^3r \; \vec{r}\times(\vec{E}\times\vec{B})$$

Let the magnetic charge be at the origin
" " electric " " " $\vec{D} = (0,0,D)$

By symmetry, $\vec{L}$ is parallel or anti-parallel to $\vec{D}$

$$\Rightarrow (\text{Mag of } \vec{L}) = \frac{1}{D}\vec{D}\cdot\vec{L} \equiv L$$

$$\vec{B} = g\frac{\hat{r}}{r^2}, \quad E = \frac{q}{4\pi\epsilon_0}\frac{\vec{r}-\vec{D}}{|\vec{r}-\vec{D}|^3}$$

$$L = g\left(\frac{q}{4\pi\epsilon_0}\right)\epsilon_0 \int d^3r \; F$$

$$F = \frac{1}{D}\vec{D}\cdot\vec{r}\times\left[(\vec{r}-\vec{D})\times\hat{r}\right]\frac{1}{|\vec{r}-\vec{D}|^3}\frac{1}{r^2}$$

$$= \frac{-1}{D}\frac{1}{r}\frac{1}{|\vec{r}-\vec{D}|^3}\underbrace{\vec{D}\cdot\hat{r}\times(D\times\hat{r})}$$

$$= \vec{D}\cdot(\vec{D} - \hat{r}\,\hat{r}\cdot\vec{D})$$

$$= D^2(1-\cos^2\theta)$$

with θ = angle between $\vec{D}$ & $\hat{r}$

$$\Rightarrow F = -\frac{D}{r}(1-\cos^2\theta)\left[r^2\sin^2\theta + (r\cos\theta - D)^2\right]^{-3/2}$$

$$L = -\frac{qg}{4\pi} D\int_0^{2\pi} d\phi \int_{-1}^{1} d\cos\theta \int_0^{\infty} dr \; r(1-\cos^2\theta)\left[r^2 - 2rD\cos\theta + D^2\right]^{-3/2}$$

Let $r = Dy$

$$\Rightarrow L = -\frac{qg}{4\pi}(2\pi)\int_{-1}^{1} d(\cos\theta)\,(1-\cos^2\theta)\int_0^{\infty} dy \; \frac{y}{(y^2-2y\cos\theta+1)^{3/2}}$$

$\Rightarrow$ Independent of D

Using the integral given with the problem,

$$L = -\frac{qg}{2}\int_{-1}^{1} d\cos\theta\,(1-\cos^2\theta)\left(\frac{1}{1-\cos\theta}\right)$$

$$= -\frac{qg}{2}\int_{-1}^{1} d(\cos\theta)\,(1+\cos\theta)$$

$$= -qg$$

$\rightarrow \vec{L}$ is antiparallel to $\vec{D}$, points from electric to magnetic charge