Columbia University
Department of Physics
QUALIFYING EXAMINATION
Monday, January 11, 2010
1:00 PM - 3:00 PM

Classical Mechanics
Section 1.

Two hours are permitted for the completion of this section of the examination. Choose **4 problems** out of the 5 included in this section. Remember to hand in **only** the 4 problems of your choice (if by mistake you hand in 5 problems, the highest scoring problem grade will be dropped). Apportion your time carefully.

Use separate answer booklet(s) for each question. Clearly mark on the answer booklet(s) which question you are answering (e.g., Section 1 (Classical Mechanics), Question 2; Section 1 (Classical Mechanics), Question 6; etc.)

Do **NOT** write your name on your answer booklets. Instead clearly indicate your **Exam Letter Code**.

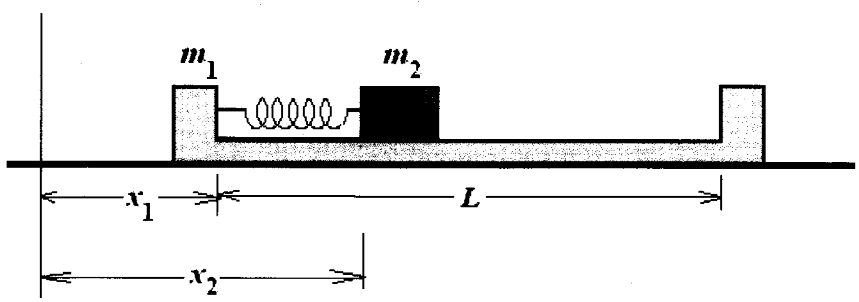
You may refer to the single handwritten note sheet on 8 $1/2 \times 11$" paper (double-sided) you have prepared on Classical Mechanics. The note sheet cannot leave the exam room once the exam has begun. This note sheet must be handed in at the end of today's exam. Please include your Exam Letter Code on your note sheet. No other extraneous papers or books are premitted.

Simple calculators are permitted. However, the use of calculators for storing and/or recovering formulae or constants is NOT permitted.

Questions should be directed to the proctor.

Good luck!!

1. A block of mass m_2 slides inside a cavity of length L inside a second block of mass m_1 which rests on a horizontal table. The masses m_1 and m_2 are connected by a massless spring with spring constant k and equilibrium length $l \ll L$. Initially both blocks are at rest and located at $x_1 = 0$ and $x_2 = l - \Delta l$ where Δl specifies the initial compression of the spring.



(a) If the mass m_1 slides without friction on the table and m_2 slides without friction on the second block, find $x_1(t)$ and $x_2(t)$ as a function of time.

(b) If the mass m_1 exerts a frictional force on m_2 proportional to their relative velocity, $F_{1\text{ on }2} = -\sigma(\dot{x}_2 - \dot{x}_1)$, again determine the resulting motion of the two masses.

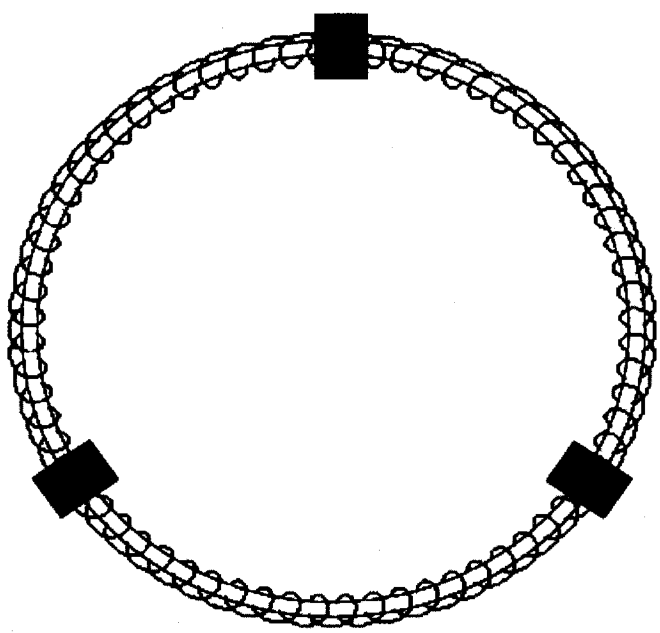
(c) If m_2 slides on m_1 without friction but m_1 experiences a similar frictional force from the table, $F_1 = -\sigma\dot{x}_1$, find the resulting complex frequencies to first order in σ assuming σ to be small. What do those frequencies imply about the resulting motion?

2. Consider the general problem of N beads of mass m, that slide frictionlessly around a fixed horizontal hoop. The beads are attached to, and spaced by, identical massless springs whose natural length is much smaller than their equilibrium length. For any given N, the spring constant is chosen such that in equilibrium the springs are under tension T. Answer the following:

   (a) Suppose $N = 2$. For $t < 0$ bead #1 is held fixed at a reference position, $\theta = 0$, and bead #2 is held fixed at $\theta = \pi + \Delta$ where $\Delta \ll \pi$. At $t = 0$ the beads are released. Find the subsequent motion of the two beads, i.e. $\theta_1(t)$ and $\theta_2(t)$.

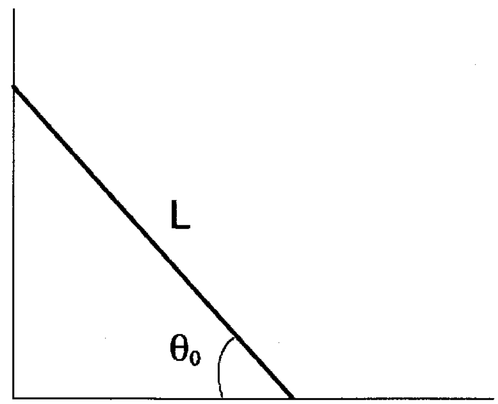
   (b) Suppose N is very large. The mass density of the beads on the hoop is μ. Estimate the two lowest frequencies for the normal modes of the system.

   (c) Suppose $N = 3$. Find the frequencies and corresponding eigenvectors of the normal modes of the system.



$N = 3$

3. A uniform ladder of mass M and length L is placed with one end against a frictionless wall and the other end on a frictionless floor. The ladder initially makes an angle θ_0 with the floor, as shown below.



The ladder is released, and slides under the influence of gravity.

(a) Write the Lagrangian for the sliding ladder as a function of θ (the angle of the ladder with respect to the floor).

(b) At what angle θ does the ladder lose contact with the wall?

(Note: The moment of inertia of a uniform rod of mass M and length L rotating about an axis through its center of mass is $I = \frac{1}{12}ML^2$)

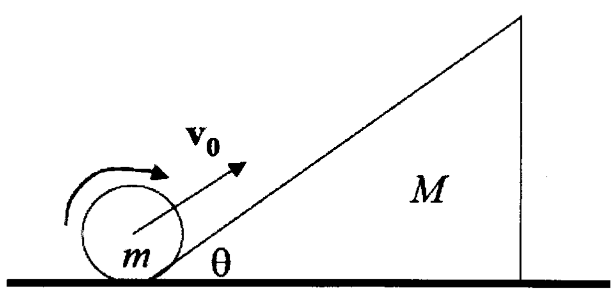
4. A cylinder of radius R and mass m rolls up an inclined plane of angle θ without slipping. The inclined plane has mass M and is free to slide along the horizontal surface without friction.

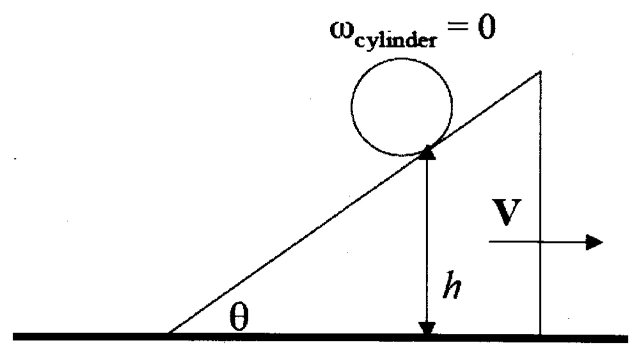
The cylinder has an initial velocity $\vec{v}_0$ up the inclined plane, and the inclined plane is initially at rest with respect to the horizontal surface.

(a) How high does the cylinder rise before it stops rotating and then starts to roll back down the inclined plane (h in the diagram)?

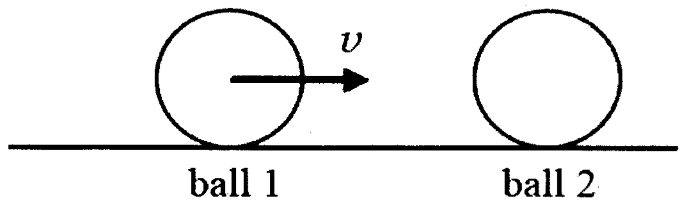
(b) At this point, what is the horizontal velocity of the cylinder and inclined plane ($\vec{V}$ in the diagram)?

(Give your answers in terms of I, R, m, M, θ, g, and v_0.)





5. Consider two identical billiard balls (spheres), each of mass M and radius R. One is stationary (ball 2) and the other rolls on a horizontal surface without slipping, with a horizontal speed v (ball 1), as shown.



Assume that all the frictional forces are small enough so as to be negligible over the time of the collision, and that the collision is completely elastic.

(a) Calculate the moment of inertia of one of the billiard balls about its center.

(b) What is the final velocity of each ball a long time after the collision? *i.e.* when each ball is rolling without slipping once more.

(c) What fraction of the initial energy is transformed into heat?

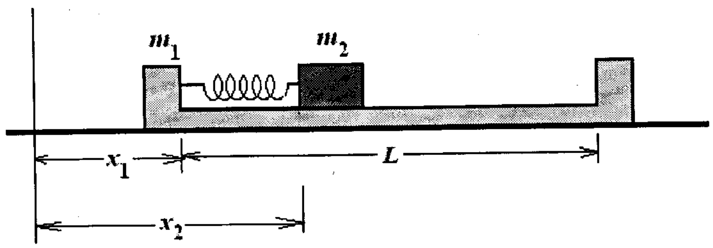
Christ
Sec. 1 MECH
1

N. Christ

November 30, 2008

Quals Problems

1. A block of mass m_2 slides inside a cavity of length L in a second block of mass m_1 which rests on a horizontal table. The masses m_1 and m_2 are connected by a massless spring with spring constant k and equilibrium length $l \ll L$. Initially both blocks are at rest and located at $x_1 = 0$ and $x_2 = l - \Delta l$ where Δl specifies the initial compression of the spring.



(a) If the mass m_1 slides without friction on the table and m_2 slides without friction on the second block find $x_1(t)$ and $x_2(t)$ as a function of time.

(b) If the m_1 exerts a frictional force on m_2 proportional to their relative velocity, $F_{1\text{ on }2} = -\sigma(\dot{x}_2 - \dot{x}_1)$, again determine the resulting motion.

(c) If m_2 slides on m_1 without friction but m_1 experiences a similar frictional force from the table, $F_1 = -\sigma\dot{x}_1$, find the resulting complex frequencies to first order in σ assuming σ to be small. What do those frequencies imply about the resulting motion?

Suggested Solution

1. (a) Start with equations of x_1 and x_2:

$$\begin{aligned} m_1\ddot{x}_1 &= -k(x_1 - x_2 + l) \\ m_2\ddot{x}_2 &= -k(x_2 - x_1 - l) \end{aligned}$$

The sum of these equations describe the free particle motion of the center of mass variable $x_{\text{cm}} = (m_1x_1 + m_2x_2)/(m_1 + m_2)$:

$$\ddot{x}_{\text{cm}} = 0$$

If the first equation is multiplied by m_2 and subtracted from the second multiplied by m_1, we find a simple harmonic equation for the variable $y = x_2 - x_1 - l$:

$$m_1m_2\ddot{y} = -(m_1 + m_2)ky \tag{1}$$

Thus, if we define $\omega_0 = \sqrt{k/\mu}$ with $\mu = m_1m_2/(m_1 + m_2)$ we have the solution:

$$\begin{aligned} y(t) &= -\Delta l \cos(\omega_0 t) \\ x_{\text{cm}} &= \frac{m_2(l - \Delta l)}{m_1 + m_2} \end{aligned}$$

(b) The extra friction force does not change the structure of the equations:

$$\begin{aligned} m_1\ddot{x}_1 &= -k(x_1 - x_2 + l) - \sigma(\dot{x}_1 - \dot{x}_2) \\ m_2\ddot{x}_2 &= -k(x_2 - x_1 - l) - \sigma(\dot{x}_2 - \dot{x}_1) \end{aligned}$$

so they are solved the same variables x_{cm} and y:

$$\begin{aligned} y(t) &= e^{-\gamma t/2}\Delta l \left\{ -\cos(\omega t) + \frac{\gamma}{2\omega}\sin(\omega t) \right\} \\ x_{\text{cm}} &= \frac{m_2(l - \Delta l)}{m_1 + m_2}, \end{aligned}$$

where $\gamma = \sigma/\mu$ and $\omega = \sqrt{\omega_0^2 - \gamma^2/4}$.

(c) The equations become less familiar if friction is introduced between the table and m_1:

$$\begin{aligned} m_1\ddot{x}_1 &= -k(x_1 - x_2 + l) - \sigma\dot{x}_1 \\ m_2\ddot{x}_2 &= -k(x_2 - x_1 - l). \end{aligned}$$

Now the center of mass motion will couple with the oscillating variables and the four frequencies present in this system of two coupled second order equations can be found by solving:

$$\begin{aligned} 0 &= \det\begin{pmatrix} -m_1\omega^2 + k + i\sigma\omega & -k \\ -k & -m_2\omega^2 + k \end{pmatrix} \\ &= m_1 m_2 \omega^4 - (m_1 + m_2)k\omega^2 + i\sigma\omega(k - m_2\omega^2). \end{aligned}$$

If $\sigma = 0$, these have the double root $\omega = 0$ and the two roots $\omega = \pm\omega_0$ corresponding to the $x_{\text{cm}}(0) + \dot{x}_{\text{cm}}(0)t$ cm mass and oscillatory motion above. These zeroth-order results can then be substituted in the above equation to find the frequencies to first order in σ:

$$\begin{aligned} \omega &= \pm\omega_0 + i\sigma\frac{m_2}{m_1(m_1 + m_2)} \\ \omega &= 0, \quad \omega = +i\frac{\sigma}{m_1 + m_2}. \end{aligned}$$

The $\omega = 0$ root corresponds to equilibrium with an arbitrary cm location, while $i\sigma/(m_1+m_2)$ describes non-oscillatory behavior with non-zero cm velocity, decreasing exponentially to zero. Finally $\pm\omega_0 + i\sigma m_2/(m_1[m_1 + m_2])$ corresponds to oscillatory motion damped by the motion of m_1.

Cole Section 1
mechanics
Problem #2

N Beads on a hoop – solutions

The displacement of bead with index i from its equilibrium position will be written ξ_i. Since the net force on a bead is zero with all of the beads at their equilibrium positions we can write the equations of motion purely in terms of the displacements ξ.

a. The spring constant k can be expressed in terms of the tension using $T = 2\pi R k$ or $k = T/2\pi R$. The equations of motion for the two beads can be written

$$\begin{aligned} mR\,\ddot{\xi}_1 &= kR\,(\xi_2 - \xi_1) - kR\,(\xi_1 - \xi_2) = 2kR\,(\xi_2 - \xi_1) \\ mR\,\ddot{\xi}_2 &= kR\,(\xi_1 - \xi_2) - kR\,(\xi_2 - \xi_1) = 2kR\,(\xi_1 - \xi_2) \end{aligned}$$

If we add and subtract the two equations of motion we obtain,

$$\begin{aligned} mR\left(\ddot{\xi}_1 + \ddot{\xi}_2\right) &= 0. \\ mR\left(\ddot{\xi}_2 - \ddot{\xi}_1\right) &= 4kR(\xi_2 - \xi_1) \end{aligned}$$

If we define $\xi_s = \xi_1 + \xi_2$ and $\xi_d = \xi_2 - \xi_1$ and simplify we obtain the two equations

$$\begin{aligned} \ddot{\xi}_2 &= 0. \\ \ddot{\xi}_d &= 4\left(\frac{k}{m}\right)\xi_d \end{aligned}$$

The first equation which describes the first normal mode of the system and has the solution $\xi_s = \xi_0 + \omega_s\, t$, describes the simultaneous motion of the beads around the circle at constant separation. The second equation which describes the second n normal mode corresponds to simple harmonic oscillation of the ξ_d coordinate with frequency $\omega_d = 2\sqrt{k/m}$. We can write the general solution of that equation, $\xi_d = A\cos\omega_d\, t + B\sin\omega_d\, t$. We can obtain the constants ξ_0, ω_s, A, and B from the initial conditions. we have

$$\begin{aligned} \xi_s(t=0) &= \xi_0 = \xi_1(t=0) + \xi_2(t=0) = \Delta/2 \\ \xi_d(t=0) &= A = \xi_2(t=0) - \xi_1(t=0) = \Delta/2 \\ \dot{\xi}_s(t=0) &= \omega_s = \dot{\xi}_1(t=0) + \dot{\xi}_2(t=0) = 0 \\ \dot{\xi}_d(t=0) &= B\omega_d = \dot{\xi}_2(t=0) - \dot{\xi}_1(t=0) = 0 \end{aligned}$$

Or more succinctly, $\xi_0 = \Delta$, $A = \Delta$, $\omega_s = 0$, $B = 0$. Now we express ξ_1 and ξ_2 in terms of ξ_s and ξ_d,

$$\xi_1 = \frac{1}{2}\left(\xi_s - \xi_d\right), \xi_2 = \frac{1}{2}\left(\xi_s + \xi_d\right)$$

with the results for $\xi_1(t)$ and $\xi_2(t)$,

$$\begin{aligned} \xi_1(t) &= \frac{\Delta}{2}\left[1 - \cos\left(\omega_d\, t\right)\right] \\ \xi_2(t) &= \frac{\Delta}{2}\left[1 + \cos\left(\omega_d\, t\right)\right] \end{aligned}$$

b. In the large-N limit the system can be thought of as effectively continuous with a wave equation for the angle-dependent displacement from equilibrium, $\xi(\theta, t)$,

$$\frac{\partial^2 \xi}{\partial t^2} = \frac{v^2}{R^2}\frac{\partial^2 \xi}{\partial \theta^2}$$

The phase velocity can be found purely through dimensional analysis – the only combination of constants in the problem that have the correct dimensions for v^2 is $v^2 = T/\mu$ (similar to waves on a string). We can write the general standing wave solution

$$\xi(\theta, t) = A \sin(kR\theta - \alpha)\cos(\omega t - \phi)$$

where α and ϕ are spatial and temporal phase angles respectively. As usual $\omega/k = v$. Now, waves that propagate on the hoop must satisfy the periodicity condition $\xi(\theta + 2\pi, t) = \xi(\theta, t)$. Thus, we are restricted to solutions where $2\pi kR = n2\pi$ or $kR = n$ where n is an integer. Thus yields values for k, $k = 1/R, 2/R, ...$ However, as with the $N = 2$ case in part a, there is a solution corresponding to no oscillation where the beads simply move around the loop at constant angular velocity. The solution corresponds to $\omega = 0$. So, strictly speaking the two lowest frequencies of motion of the systemn have $\omega = 0$ and $\omega = v/R$.

c. The equations of motion take the form

$$\ddot{\xi}_i = -\frac{k}{m}\left(2\xi_i - \xi_{i-1} - \xi_{i+1}\xi\right) \equiv \omega_0^2\left(2\xi_i - \xi_{i-1} - \xi_{i+1}\xi\right)$$

with i cyclic: $i = 0 \rightarrow i = 3$and $i = 4 \rightarrow i = 1$. hewre we have defined with $\omega_0 \equiv \sqrt{k/m}$. If we assume the existence of normal mode solutions to the motion of the form $U = A\cos(\omega\, t - \phi)$ with $\xi_i = C_i U$ and substitute into the equations of motion we obtain an eigenvalue equation

$$\begin{bmatrix} \omega^2 - 2\omega_0^2 & \omega_0^2 & \omega_0^2 \\ \omega_0^2 & \omega^2 - 2\omega_0^2 & \omega_0^2 \\ \omega_0^2 & \omega_0^2 & \omega^2 - 2\omega_0^2 \end{bmatrix}\begin{bmatrix} C_1 \\ C_2 \\ C_3 \end{bmatrix} = \omega_0^2 \begin{bmatrix} r^2 - 2 & 1 & 1 \\ 1 & r^2 - 2 & r^2 \\ 1 & 1 & r^2 - 2 \end{bmatrix}\begin{bmatrix} C_1 \\ C_2 \\ C_3 \end{bmatrix} = 0$$

$r = \omega/\omega_0$. Applying the usual requirement on the determinant (zero)

$$Det \begin{bmatrix} r^2 - 2 & 1 & 1 \\ 1 & r^2 - 2 & 1 \\ 1 & 1 & r^2 - 2 \end{bmatrix} = 0$$

we obtain the characteristic equation

$$\left(r^2 - 2\right)\left(\left(r^2 - 2\right)^2 - 1\right) - \left(r^2 - 2 - 1\right) + \left(1 - \left(r^2 - 2\right)\right) = 0$$

Simplifying, we can write the characteristics equation

$$\left(r^2 - 2\right)^3 - 3\left(r^2 - 2\right) + 2 = 0$$

Expanding out all the terms and cancelling where appropriate we obtain

$$r^6 - 6r^4 + 9r^2 = r^2\left(r^2 - 3\right)^2 = 0$$

with the solutions $r^2 = 0$ and (degenerate) $r^2 = 3$ (taking only the positive root for solutions to normal mode motion. The $r^2 = 0$ solution corresponds to no oscillation. The resulting equation(s) for the (unnormalized) eigenvector taking $C_1 = 1$ are

$$C_2 + C_3 = 2, -2C_2 + C_3 = -1$$

which give as solutions, $C_2 = 1$ and $C_3 = 1$ for a normalized eigenvector,

$$C = \sqrt{\frac{1}{3}} \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$$

Clearly this solution corresponds to the simultaneous motion of the beads around the hoop. Now consider the degenerate solution $r^2 = 3$ which means $\omega = \sqrt{3}\omega_0$. We obtain a redundant equation for the eigenvectors, $C_1 + C_2 + C_3 = 0$. The redundancy (due to the degeneracy which, in turn results from the symmetry of the problem under cyclic permutation of the indices) means that we have freedom in choosing the remaining two eigenvectors as long as they are orthogonal. One valid choice based on intuition about how normal modes work is to have one bead fixed and the others to oscillate with opposite phase, namely

$$C = \sqrt{\frac{1}{2}} \begin{bmatrix} 1 & -1 & 0 \end{bmatrix}$$

Giving a final eigenvector

$$C = \sqrt{\frac{2}{3}} \begin{bmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} \end{bmatrix}$$

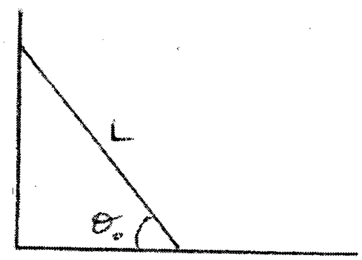
Dodd

Sec. 1 MECH

#3

2010 Quals Question: Mechanics (Dodd)

A uniform ladder of mass M and length L is placed with one end against a frictionless wall and the other end on a frictionless floor. The ladder initially makes an angle θ_0 with the floor, as shown below.



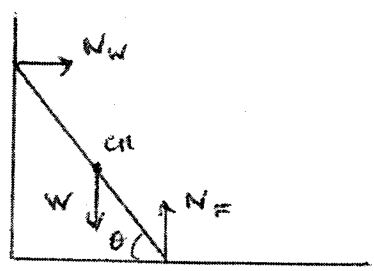
The ladder is released, and slides, under the influence of gravity.

a). Write the Lagrangian for the sliding ladder as a function of θ (the angle of the ladder with respect to the floor).

b). At what angle θ does the ladder lose contact with the wall?

(The moment of inertia of a uniform rod of mass M and length L rotating about an axis through its center of mass is $I = \frac{1}{12}ML^2$.)

Solution:



a). Denoting the center of mass coordinates of the ladder by (x_{CM}, y_{CM}), then the Lagrangian is:

$$L = T - V$$

where:

$$T = \frac{1}{2}M(\dot{x}_{CM}^2 + \dot{y}_{CM}^2) + \frac{1}{2}I_{CM}\dot{\theta}^2$$

with $I_{CM} = \frac{1}{12}ML^2$, and $(\dot{x}_{CM}^2 + \dot{y}_{CM}^2) = \left(\frac{L}{2}\right)^2 \dot{\theta}^2$, so:

$$T = \frac{1}{2}M\left(\frac{L}{2}\right)^2 \dot{\theta}^2 + \frac{1}{2}\left(\frac{1}{12}ML^2\right)\dot{\theta}^2 = \frac{1}{6}ML^2\dot{\theta}^2$$

and:

$$V = Mg\left(\frac{L}{2}\right)sin\theta$$

so the Lagrangian can be written:

$$\underline{L = \frac{1}{6}ML^2\dot{\theta}^2 - \frac{1}{2}MgLsin\theta}$$

b). The equation of motion, via the Lagrange equation:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0$$

is:

$$\frac{1}{3}ML^2\ddot{\theta} + \frac{1}{2}MgLcos\theta = 0$$

i.e.

$$\frac{1}{3}L\ddot{\theta} + \frac{1}{2}gcos\theta = 0$$

As the ladder slides, energy E is conserved, where:

$$E = T + V = \frac{1}{6}ML^2\dot{\theta}^2 + \frac{1}{2}MgLsin\theta$$

and we know the total energy from the initial condition (with no velocity), viz.:

$$E_0 = \frac{1}{2}MgLsin\theta_0$$

giving:

$$\frac{1}{6}ML^2\dot{\theta}^2 = \frac{1}{2}MgL(sin\theta_0 - sin\theta)$$

i.e.

$$\frac{1}{3}L\dot{\theta}^2 = g(sin\theta_0 - sin\theta)$$

Writing the center of mass coordinates in terms of L and θ: $x_{CM} = \left(\frac{L}{2}\right)cos\theta$, and $y_{CM} = \left(\frac{L}{2}\right)sin\theta$, and looking at horizontal forces, gives:

$$N_W = M\ddot{x}_{CM} = M\left(\frac{L}{2}\right)\left(-\dot{\theta}^2 cos\theta - \ddot{\theta}sin\theta\right)$$

At the point at which the ladder breaks contact with the wall, $N_W = 0$, and so:

$$-\dot{\theta}^2 cos\theta - \ddot{\theta}sin\theta = 0$$

i.e.

$$\ddot{\theta} = -\dot{\theta}^2 cot\theta$$

Substituting in the equation of motion gives:

$$\frac{1}{3}L\left(-\dot{\theta}^2 cot\theta\right) + \frac{1}{2}gcos\theta = 0$$

i.e.

$$\frac{1}{3}L\dot{\theta}^2 = \frac{1}{2}gsin\theta$$

Lastly, substitute into the energy conservation equation, to give:

$$\frac{1}{2}gsin\theta = g(sin\theta_0 - sin\theta)$$

i.e.

$$sin\theta = \frac{2}{3}sin\theta_0$$

and:

$$\underline{\theta = sin^{-1}\left(\frac{2}{3}sin\theta_0\right)}$$

Mech. Shaevitz
Sec. 1 MECH

Quals Problem 1 – Mechanics

M. Shaevitz
Fall, 2009

#4

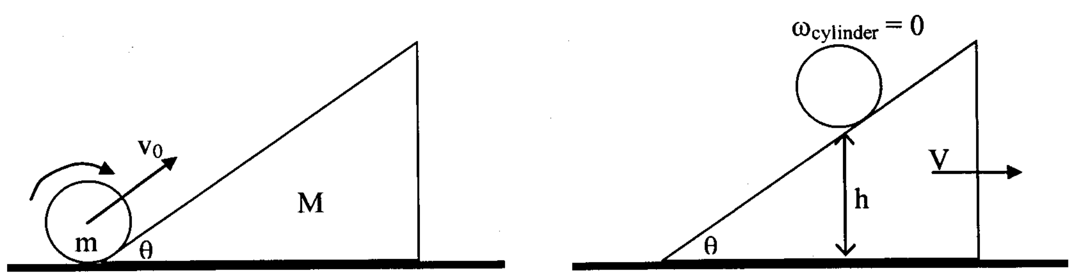
A cylinder of radius **R** and mass **m** rolls up an inclined plane of angle **θ** without slipping. The inclined plane has mass **M** and is free to slide along the horizontal surface without friction.

The cylinder has an initial velocity up the incline of $\mathbf{v_0}$ and the inclined plane is initially not moving with respect to the horizontal surface.

a) How high does the cylinder rise before it stops rotating and then starts to roll back down the inclined plane (**h** in the diagram)?

b) At this point, what is the horizontal velocity of the cylinder and inclined plane (**V** in the diagram)?

(Give your answers in terms of R, m, M, θ, g, and v_0.)



Freshman Physics Solution:

Since No external forces in the x-direction, the horizontal momentum is conserved. (The friction between cylinder and ramp is internal.)

Cons Mom: $m \dot{x}_0 \cos\theta = (m+M)\dot{X}_{TOP}$

Cons. Energy: $\frac{1}{2} m \dot{x}_0^2 + \frac{1}{2}\left(\frac{I}{R^2}\right)\dot{x}_0^2 = \frac{1}{2}(m+M)\dot{X}_{TOP}^2 + mgh$

$$\dot{X}_{TOP} = \frac{m\cos\theta}{(m+M)}\dot{x}_0$$

$$\left(m + \frac{I}{R^2}\right)\dot{x}_0^2 = (m+M)\frac{m^2\cos^2\theta}{(m+M)^2}\dot{x}_0^2 + 2mgh$$

a) $$h = \left(m + \frac{I}{R^2} - \frac{m^2\cos^2\theta}{(m+M)}\right)\frac{1}{(2mg)}\dot{x}_0^2$$

b) $$\dot{X}_{TOP} = \frac{m\cos\theta}{(m+M)}\dot{x}_0$$

Lagrangian Solution: Use x = distance up incline + X position of incline

$$v_{cylinder} = (\dot{x}\cos\theta + \dot{X},\ \dot{x}\sin\theta)$$

$$v_{plane} = \dot{X} \qquad \dot{\theta} = \dot{x}/R$$

$$T = \frac{1}{2}m\left[(\dot{x}\cos\theta + \dot{X})^2 + \dot{x}^2\sin^2\theta\right] + \frac{1}{2}M\dot{X}^2 + \frac{1}{2}I\left(\frac{\dot{x}}{R}\right)^2$$

$$= \frac{1}{2}\left(m + \frac{I}{R^2}\right)\dot{x}^2 + \frac{1}{2}(m+M)\dot{X}^2 + m\dot{x}\dot{X}\cos\theta$$

$$U = mgx\sin\theta$$

$$\mathcal{L} = T - U = \frac{1}{2}\left(m + \frac{I}{R^2}\right)\dot{x}^2 + \frac{1}{2}(m+M)\dot{X}^2 + m\dot{x}\dot{X}\cos\theta - mgx\sin\theta$$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right) - \frac{\partial \mathcal{L}}{\partial x} = 0 = \left(m + \frac{I}{R^2}\right)\ddot{x} + m\ddot{X}\cos\theta + mg\sin\theta = 0$$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{X}}\right) - \frac{\partial \mathcal{L}}{\partial X} = 0 = (m+M)\ddot{X} + m\ddot{x}\cos\theta = 0$$

$$\ddot{X} = -\frac{\ddot{x}m\cos\theta}{m+M}$$

$$\left(m + \frac{I}{R^2}\right)\ddot{x} - \frac{m^2\cos^2\theta}{(m+M)}\ddot{x} = -mg\sin\theta$$

$$\ddot{x}\left(m + \frac{I}{R^2} - \left(\frac{m^2}{m+M}\right)\cos^2\theta\right) = -mg\sin\theta$$

Constant acceleration preomblem with

$$M_{eff} = \left(m + \frac{I}{R^2} - \left(\frac{m^2}{m+M}\right)\cos^2\theta\right) \Rightarrow a = \frac{-mg\sin\theta}{M_{eff}}$$

a) At top $\dot{x}_{TOP} = 0$, $x = \frac{h}{\sin\theta}$ Initial $x = 0$ $\dot{x} = \dot{x}_0$

$X = 0$ $\dot{X} = 0$

$$\dot{x}_{TOP}^2 = \dot{x}_0^2 + 2a x_{TOP} = \dot{x}_0^2 - \frac{2mg\sin\theta}{M_{eff}} x_{TOP}$$

$$\Rightarrow h = x_{TOP}\sin\theta = \frac{(M_{eff})\,\dot{x}_0^2}{2mg} \quad \text{(same as before)}$$

b) From $(m+M)\ddot{X} = -\ddot{x}m\cos\theta$

$$\int_0^{\dot{X}_{TOP}} (m+M)\,d\dot{X} = \int_{+\dot{x}_0}^{0} -m\cos\theta\,d\dot{x} \Rightarrow (m+M)\dot{X}_{TOP} = m\dot{x}_0\cos\theta$$

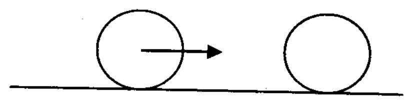
Again $\dot{X}_{TOP} = \left(\frac{m}{m+M}\right)\cos\theta\,\dot{x}_0$

Mechanics Tuts
Sec 1 MECH
#5

Quals – Mechanics Question – Tuts – 11/25/09

QUESTION

Consider two identical billiard balls (spheres), each of mass M and radius R. One is stationary (ball 2) and the other rolls on a horizontal surface without slipping with a horizontal speed v (ball 1), as shown. Assume that all the frictional forces are small enough so as to be negligible over the time of the collision, and that the collision is completely elastic.



A. Calculate the moment of inertia of one of the billiard balls about its center.
B. What is the final velocity of each ball a long time after the collision? *i.e.* when each ball is rolling without slipping once more.
C. What fraction of the initial energy is transformed into heat?

SOLUTION

Part A

$$I = \int r^2 dm$$

$$I = \int x^2 2\pi\rho x\, 2\sqrt{R^2 - x^2}\, dx$$

$$= 4\pi\rho \int x^3 \sqrt{R^2 - x^2} dx$$

$$= 4\pi\rho R^5 \int \left(\frac{x}{R}\right)^3 \sqrt{\left(1 - \left(\frac{x}{R}\right)^2\right)}\, dx$$

$$= 4\pi\rho R^5 \int \left(\frac{x}{R}\right)^2 \sqrt{1 - \left(\frac{x}{R}\right)^2} \frac{1}{2} d\left(\frac{x}{R}\right)^2$$

$$let\ y = \left(\frac{x}{R}\right)^2$$

$$= 2\pi\rho R^5 \int y \sqrt{1-y} dy$$

$$integrate\ by\ parts\ u = y, dv = \sqrt{1-y}\ hence\ du = dy\ and\ v = -\frac{2}{3}(1-y)^{\frac{3}{2}}$$

$$I = 2\pi\rho R^5 [-\frac{2}{3}(1-y)^{\frac{3}{2}} y - \int (-\frac{2}{3}(1-y)^{\frac{3}{2}} dy)$$

$$= 2\pi\rho \backslash R^5 \left[-\frac{2}{3}\left(\frac{x}{R}\right)^2 \left(1 - \left(\frac{x}{R}\right)^2\right)^{\frac{3}{2}} - \frac{4}{15}\left(1 - \left(\frac{x}{R}\right)^2\right)^{\frac{5}{2}}\right] \text{from x=0 to R}$$

$$\therefore I = \left(\frac{2}{5}\right) MR^2$$

Part B

Just before the collision

$v_1 = v$

$v_2 = 0$

$$\omega_1 = \frac{v_1}{R} = \frac{v}{R}$$

$\omega_2 = 0$

Just after the collision (and since friction is negligible during collision and it is elastic)

$v_1' = 0$

$v_2' = v$

And the angular momenta about the center of each ball are conserved, hence

$$\omega_1' = \omega_1 = \frac{v}{R}$$

And $\omega_2' = \omega_2 = 0$

Now, if we look "a long time later" (where the balls are rolling without slipping), then for each ball we can use angular momentum conservation about a fixed point on the surface where the balls rolls then for ball 1

$$L_1 = MRv_1' + I\omega_1' = MRv_1'' + I\omega_1''$$

$$= v_1''\left(MR + \frac{I}{R}\right)$$

Or replacing in for L

$$I\omega_1' = \frac{Iv}{R} = v_1''\left(MR + \frac{I}{R}\right)$$

$$v_1'' = \frac{vI}{I + MR^2} = \frac{v\left(\frac{2}{5}\right)MR^2}{\left(\frac{7}{5}\right)MR^2}$$

$$v_1'' = \left(\frac{2}{7}\right)v$$

Similarly for ball 2 we arrive at

$$v_2'' = \left(\frac{5}{7}\right)v$$

Part C

$$KE_{initial} = \left(\frac{1}{2}\right)Mv_1^2 + \frac{1}{2}I\omega_1^2$$

$$= \frac{1}{2}Mv^2 + \frac{1}{2}MR^2\left(\frac{v}{R}\right)^2 = \frac{1}{2}Mv^2\frac{7}{5}$$

$$KE_{final} = \frac{1}{2}\frac{7}{5}M\left(\left(\frac{2}{7}v\right)^2 + \left(\frac{5}{7}v\right)^2\right) = \frac{1}{2}\times\frac{7}{5}Mv^2\frac{29}{49}$$

So the energy lost to friction is

$$KE_{initial} - KE_{final} = \frac{1}{2}\times\frac{7}{5}\,Mv^2\frac{20}{49}$$

So the fraction converted to heat is

$$\frac{KE_{initial} - KE_{final}}{KE_{initial}} = \frac{20}{49}$$