Columbia University
Department of Physics
QUALIFYING EXAMINATION
Friday, January 16, 2009
3:10 PM - 5:10 PM

General Physics (Part II)
Section 6.

Two hours are permitted for the completion of this section of the examination. Choose 4 problems out of the 6 included in this section. (You will not earn extra credit by doing an additional problem). Apportion your time carefully.

Use separate answer booklet(s) for each question. Clearly mark on the answer booklet(s) which question you are answering (e.g., Section 6 (General Physics), Question 3; Section 6 (General Physics), Question 5, etc.)

Do **NOT** write your name on your answer booklets. Instead clearly indicate your **Exam Letter Code**.

You may refer to the single handwritten note sheet on 8 ½ x 11" paper (double-sided) you have prepared on General Physics. The note sheet cannot leave the exam room once the exam has begun. This note sheet must be handed in at the end of today's exam. Please include your Exam Letter Code on your note sheet. No other extraneous papers or books are permitted.

Simple calculators are permitted. However, the use of calculators for storing and/or recovering formulae or constants is NOT permitted.

Questions should be directed to the proctor.

Good luck!!

1. Provide an example of one phenomenon in solid state physics where a "quantum" aspect is important. What is the corresponding observable property which can be estimated using the uncertainty principle? Explain how the observable value can be derived using the uncertainty principle, and describe how you would experimentally measure this observable quantity.

2. You have to measure the index of refraction of a transparent liquid in the hope that it will help you identify the substance. You have plenty of the liquid, a 632nm red laser pointer and a thin walled rectangular glass container ($n_G = 1.5$). Paper, pencil, protractors and rulers are also available.

Describe three different ways to determine (measure) the index of refraction of the liquid. Try to qualitatively compare their accuracy.

3. Bound states composed of a heavy quark/anti-quark pair can be well approximated by non-relativistic quantum mechanics. Investigations of such systems provided early insight into the force between quarks. Consider the currently known spectrum of states for charmonium, made of a c quark and a c anti-quark and denoted by J/ψ, and bottomonium, made of a b and an anti-b and given by Υ.

The figure gives current values for the J/ψ and Υ masses for states labeled by quantum numbers for radial eigenfunctions and orbital angular momentum. The question marks on the figure are for states whose identity is not fully confirmed.

(a) It is known that the S states in the figure have total angular momentum $J = 1$. What is the orbital angular momentum and spin contribution to the total J?

(b) Is a $J = 0$ and $L = 0$ state allowed?

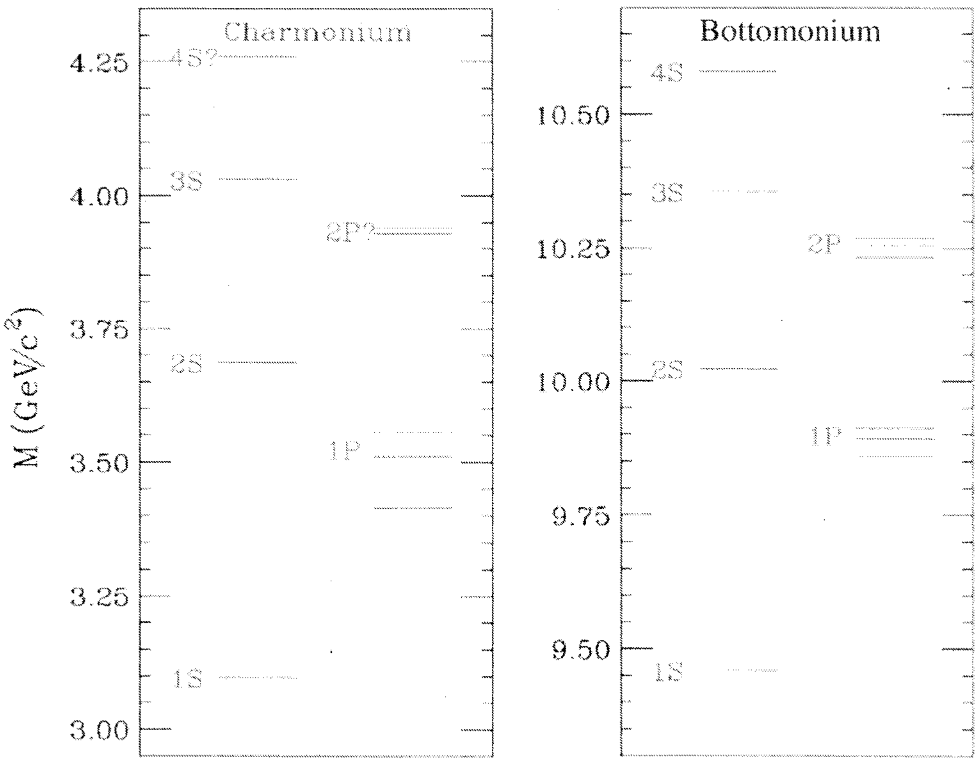
(c) If charmonium and bottomonium bound states are produced in e^+e^- annihilation, will it be easier to produce $J = 0$ states or $J = 1$ states?

(d) What values of J are present in the $1P$ states?

(e) Notice that the splittings between the states are very similar, in spite of the fact that the mass of the Υ is about 2.5 times the mass of the J/ψ. This means that the b quark is much heavier than the c quark, provided the binding energy of the two systems are not wildly different. In the radial part of the Schrödinger equation

$$Hu(r) = \left[-\frac{\hbar^2}{2mr^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{\vec{L}\cdot\vec{L}}{2mr^2} + V(r)\right]u(r) = Eu(r)$$

the mass enters and should also generally enter the expression for the eigenvalues. Show that a simple potential of the form $V(r) = \alpha \ln(r/r_0)$ leads to mass independent splittings between the states.

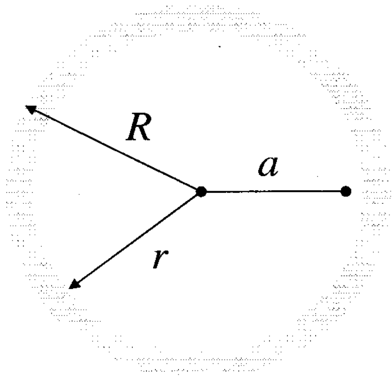


Charmonium
4S?
3S
2P?
2S
1P
1S
4.25
4.00
3.75
3.50
3.25
3.00
M (GeV/c²)
Bottomonium
4S
3S
2P
2S
1P
1S
10.50
10.25
10.00
9.75
9.50

4. A glass spherical shell of thickness 1 cm and inner radius $r = 6.5\,\text{cm}$ ($n_2 = 1.5$) is filled with a non-absorbing liquid of $n_1 = 1.6$. A point source of light is installed at $a = 6\,\text{cm}$ from the center of the sphere.

What fraction of the light gets out of the sphere?

Hint: Neglect absorption. If you have to use approximations, please explain them precisely. Also, $n_{air} = 1$.



SOLUTIONS Section 6
General Part II

Uemura · quantum exp.
Problem #1

2009 Quals: Problem and some examples of solution: by Tomo Uemura

Provide an example of one phenomenon in solid state physics where a "quantum" aspect is important. What is the corresponding observable property which can be estimated using the uncertainty principle. Explain how the observable value can be derived using the uncertainty principle, and describe how you would experimentally measure this observable quantity.

Examples of possible solutions:

(1) Bose Condensation Temperature and Fermi Temperature:

Bose condensation can be visualized as a phenomenon which occurs when the thermal wave-length (spread of wave function in real space) of a boson becomes comparable to the inter-particle distance (distance to nearest neighbour boson).

At a given temperature T, the kinetic energy of a boson in a 3-dimensional gas is given as

$$ (3/2)*k_{B}T = (1/2)p^{2}/m $$, where m is the mass of a boson.

The thermal wavelength is defined as $\lambda_{th} = h/p$
Let us assume the particle density of Bose gas as n. Then the interparticle distance is $n^{-1/3}$

This logic is the same as treating the momentum p as Δp, and λ_{th} as Δx, and then applying uncertainty principle (forget about ambiguity of 2π).

These treatments would lead to the condensation temperature T_{BE} as $k_{B}T_{BE} = (h^{2}) (n^{2/3})/m$

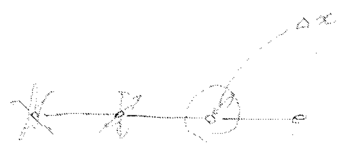
Similar argument can be used to make a crude estimate of Fermi temperature. To satisfy Pauli principle of fermions, with the particle density n, one fermion occupies the volume of $1/n$. If we assign a cube for this volume, the edge of the cube Δx will become $n^{-1/3}$. To confine each electron in such a small region, we need to give a momentum $\Delta p = h / \Delta x$, according to uncertainty principle. Then the kinetic energy of each fermion would become
$(1/2)(\Delta p)^{2}/m = (h^{2}/2)(n^{2/3})/m \sim k_{B}T_{F}$

This is a very crude way to estimate the Fermi energy $k_{B}T_{F}$. This argument shows that T_{F} and T_{BE} have the comparable magnitudes with n as the particle density, and m as the mass. For non-interacting Bose gas and Fermi gas, better calculation really shows that T_{BE} is very close to T_{F} if the densities and masses of fermions and bosons are equivalent. For BE-BCS crossover, we usually consider two fermions to form a boson with twice the fermion mass, T_{BE} becomes about ¼ T_{F}. This relationship was established by the BE-BCS crossover experiments for cold atoms.

The Bose condensation temperature in superfluid He, for example, can be determined by measuring viscosity using tortional oscillator. Fermi temperature of electron gas is usually determined by two independent measurements of n and m, and/or their combinations. n can be measured by Hall effect, while m can be measured by cyclotron resonance mass. Combination of n and m can be measured by specific heat linear term at low temperatures (Sommerfeld Constant), Pauli susceptibility, Plasma oscillation frequency, etc.

(2) Debye Waller Factor

At finite temperature T, atoms of a crystal lattice are fluctuating around their equilibrium positions due to thermal effect (in some cases like He or other light atoms, quantum fluctuations are also important). Let us express the amplitude of this fluctuation by using the average displacement Δx.



Classically

$$\frac{1}{2} k (\Delta x)^2 \sim k_B T$$

$$\exp\left(-\frac{r^2}{2(\Delta x)^2}\right) \quad \cdots \text{ real space}$$

This gives "spread in real space", and thus results in slow decay of Bragg peak intensities with increasing momentum transfer q in X-ray or neutron scattering experiments. This phenomenon can be derived by Fourier transforms of real-space correlation function. However, the essence comes from the uncertainty principle
$\Delta x * \Delta p \sim \hbar$

Usually, the decreasing Bragg peak intensity with increasing momentum transfer q is fitted by a Gaussian decay $I(q) \sim \exp[(-1/2)(q^{2}/(\Delta p)^{2}]$, which corresponds to the Gaussian spread in real space with the probability of displacement
$P(x) = \exp [(-1/2) x^{2}/(\Delta x)^{2}]$

We can connect this phenomenon with thermal excitations of phonons in the following way:

$k_{B}T \sim (1/2)C(\Delta x)^{2}$, where C is the spring constant of lattice vibration.

The frequency for this vibration is $\omega \sim \sqrt{C/M}$ where M is the atomic mass.

The Debye temperature Θ_{D} is proportional to the vibration energy $\hbar\omega \sim \Theta_{D}$

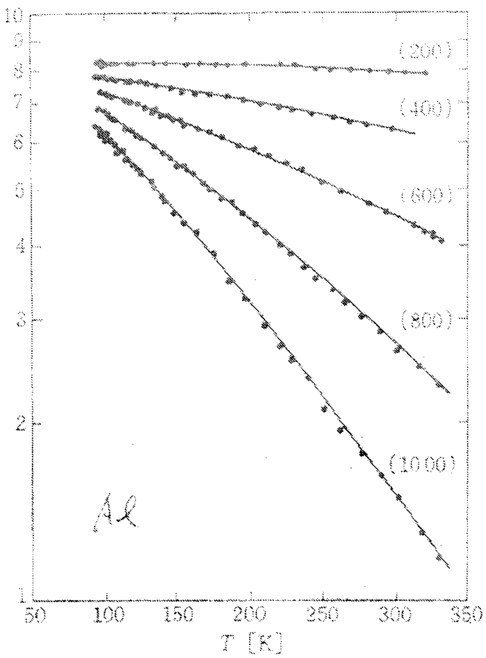
These relationships lead to

$[1/(\Delta p)^{2}] \sim (\hbar)^{2} T / [M k_{B} \Theta_{D}^{2}]$,

as found by Debye.

One can verify this relationship by performing measurements of phonon dispersion (to determine Debye temperature) and the momentum-transfer dependence of the Bragg Peak intensity I (q) (to determine Δp) by neutron and/or X-ray scattering, temperature T by a thermometer, and the atomic mass M from chemistry.

I attach an example of T- and q- dependences of the scattering intensity measured in Aluminum by X-rays. The vertical axis represents relative scattering intensity of several Bragg Peaks.



SOLUTIONS Section 6
General Part II

Mark Exp. Techniques
Sec 6 GEN II
Problem # 2
Marka

Problem:

You have to measure the index of refraction of a transparent liquid (n_L) in a hope that it will help you identify the substance. You have plenty of the liquid, a 632nm red laser pointer and a thin walled rectangular glass container (n_G=1.5). Paper, pencil, protractors and rulers are also available.

Describe three different ways to determine (measure) the index of refraction of the liquid. Try to qualitatively compare their accuracy.

Hints of solutions:

This problem is intentionally open ended. It is hoped that many ingenious and precise measurements shall be invented. We mention three possible ways to measure the index of refraction of the liquid (n). (As the wall of the container is very thin its existence is neglected for simplicity. Of course, precise measurements can take that into account.)

1. Fix the laser pointer above the container and ensure a large enough angle of incidence (to the vertical). Mark the point at the bottom of the empty container where the laser hits it ('***A***'). Fill the container with the liquid and measure the distance between ***A*** and the new laser spot ('***B***') at the bottom. From the index of refraction of the air (n_{AIR}=1), the angle of incidence, d_{AB}, and the height of the liquid one can compute n_L.

2. Aim the laser pointer to the surface of the liquid from below (i.e., through the side of the container. Adjust the angle of the laser pointer until the threshold of total internal reflection (i.e., no outgoing beam). From n_{AIR}, and the angle of the laser to the horizontal n_L can be computed.

3. Place a sheet of paper at the bottom of the container (under the liquid). Shine the laser pointer to the middle of the paper from below and observe the light pattern on the paper (opaque screen). You will see a bright point in the middle (the laser) a darker ring around and a larger bright ring around. The inner edge of the bright ring corresponds to the onset of total internal reflection on the liquid-air interface. From n_{AIR}, the thickness of the liquid, and the inner radius of the bright ring one can compute n_L.

Naturally, there are a number of other possibilities. Find some! The methods listed above are quite rudimentary and their accuracy should be comparable (not too good).

Sec 6 GEN II
Problem # 3
Mawhinney

Solutions

1. For an S state, $L = 0$, so the only way to get $J = 1$ is to have the total spin $S = 1$. Since a quark and an anti-quark are not identical particles, they can be in a symmetric spin state without violating the Pauli exclusion principle.

2. Yes. $S = 0$ is allowed, since the quark and anti-quark can have their spins opposite.

3. It is easier to produce $J = 1$ states, since the photon is what couples $e^+ e^-$ to quark/anti-quark states and the photon has spin 1.

4. The 1P states have $L = 1$ and the spins can be $S = 0, 1$, so $J = 0, 1$ and 2 are allowed.

5. Let $\rho \equiv m^{1/2} r$ and write the Schrodinger equation in terms of ρ. Now, the only m dependence in the Schrodinger equation is from

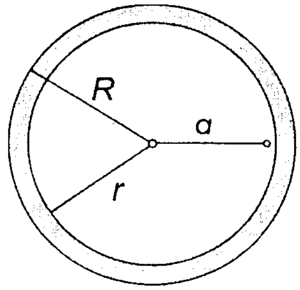
$$V(r) = \alpha \ln(\rho) - \alpha \ln(m^{1/2} r_0) \tag{2}$$

and the second term is a constant. Thus it shifts all the energy eigenvalues, but the splitting between eigenvalues is independent of mass.

Marka – Optics
Sec 6 gen II
Problem # 4

Problem:

A glass spherical shell of thickness 1cm and inner radius of r=6.5cm (n_2=1.5) is filled with a non-absorbing liquid of n_l=1.6. A point source of light is installed at a=6cm from the center of the sphere.

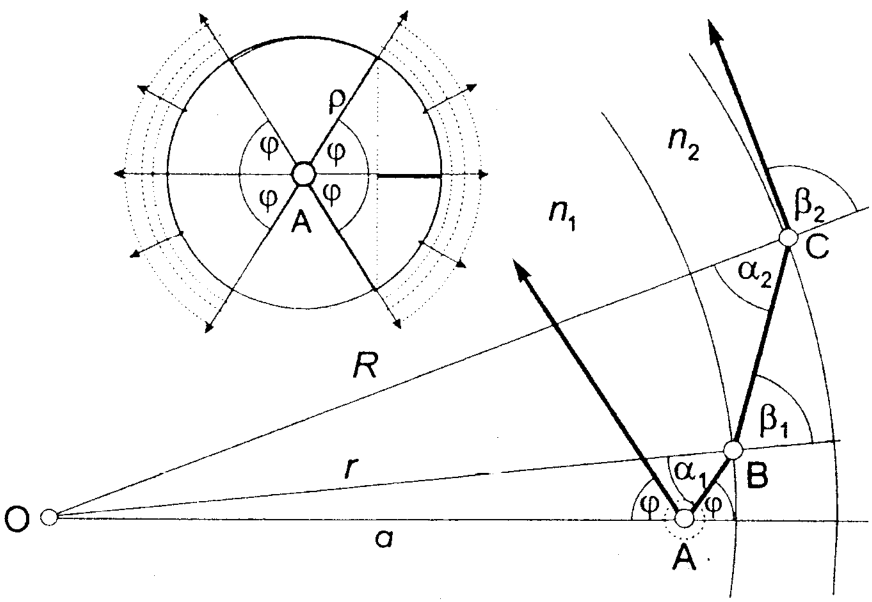


What fraction of the light gets out of the sphere?

(Hint: Neglect absorption. If you have to use approximations, please explain them precisely. Also, n_{AIR}=1.)

Hint of a solution:

Please study the geometry of the system:



We give an approximate solution to the problem. Assuming that the absorption is negligible, we we do not have to track the large number of reflections and long travel inside the materials. Due to the symmetry of the system it is reasonable to state that light-rays where $\beta_2 \leq 90°$ are the ones that will (eventually) leave the system. Therefore one needs to determine the half-opening-angle (φ) of the cone of light that gets out of the system ($\beta_2 = 90°$).

Nicolis - Order of magnitude
Sec 6 GEN II
Problem # 5

Quals 2008-09

1 General: order of magnitude estimates

A perfect fluid is described by the continuity and Euler equations, which govern the time-evolution of the density- and velocity-fields $\rho(\vec{x}, t)$, $\vec{v}(\vec{x}, t)$:

$$\dot{\rho} + \vec{\nabla} \cdot (\rho \vec{v}) = 0 \tag{1}$$

$$\dot{\vec{v}} + (\vec{v} \cdot \vec{\nabla})\vec{v} = -\frac{1}{\rho}\vec{\nabla}p \ . \tag{2}$$

Here we assume that the pressure p is a given function of ρ:

$$p = p(\rho) \ . \tag{3}$$

It is well known that the above equations, when *linearized* about the homogeneous, static background configuration

$$\rho = \rho_0 \ , \qquad \vec{v} = 0 \ , \qquad p = p_0 \equiv p(\rho_0) \ , \tag{4}$$

admit wave-like solutions, which describe sound waves. Sound waves are just small density perturbations

$$\rho(\vec{x}, t) = \rho_0 + \delta\rho(\vec{x}, t) \ , \qquad \delta\rho(\vec{x}, t) = \varepsilon \, e^{i(\vec{k}\cdot\vec{x} - \omega t)} + \text{c.c.} \tag{5}$$

that propagate at the speed

$$c_s = \sqrt{\left.\frac{dp}{d\rho}\right|_{\rho=\rho_0}} \ . \tag{6}$$

1. Without keeping track of the vector structure and of order one coefficients, estimate from the equations of motion (??, ??) the amplitude of the fluctuations in $\vec{v}$ and p corresponding to the sound wave (??). Express your results in terms of ε, k, and c_s.

2. Estimate the regime of validity of the linear approximation, that is find a condition on the quantities ε, k, and c_s that tells you when you can trust the linear approximation. Do you find your result reasonable, on physical grounds?

Solution

1. The perturbed values for ρ, $\vec{v}$, and p are

$$\rho = \rho_0 + \delta\rho \ , \qquad \vec{v} = \delta\vec{v} \ , \qquad p = p_0 + \delta p \ . \tag{7}$$

However by assumption p is purely a function of ρ, so that at linear order in $\delta\rho$ we have

$$\delta p = \left.\frac{dp}{d\rho}\right|_{\rho=\rho_0} \delta\rho \sim c_s^2 \, \varepsilon \ . \tag{8}$$

To estimate $\delta\vec{v}$ we have to linearize the equations of motion in $\delta\rho$, $\delta\vec{v}$, and δp. Using that for wave solution behaving like $e^{i(\vec{k}\cdot\vec{x}-\omega t)}$

$$\nabla \sim i\vec{k}\,, \qquad \frac{\partial}{\partial t} \sim -i\omega \sim -ic_s k\,, \tag{9}$$

and neglecting the vector structure and numerical factors, we get

$$c_s k\,\varepsilon + k\rho_0\,\delta v \sim 0 \tag{10}$$

$$c_s k\,\delta v \sim -\frac{1}{\rho_0} k\,\delta p \tag{11}$$

which both yield

$$\delta v \sim c_s \frac{\varepsilon}{\rho_0}\,. \tag{12}$$

2. To determine the regime of validity of the linear approximation, we have to estimate the size of the terms left out by the latter—the non-linear terms in the equations of motion.

   In eq. (??) there is only one such term:

$$\vec{\nabla}\cdot(\delta\rho\,\delta\vec{v}) \sim k\,\varepsilon\, c_s\frac{\varepsilon}{\rho_0}\,, \tag{13}$$

   whereas the linear pieces are of order $c_s k\varepsilon$.

   In eq. (??) we have one source of non-linearities on the l.h.s.:

$$(\delta\vec{v}\cdot\vec{\nabla})\delta\vec{v} \sim k\,c_s^2\,\frac{\varepsilon^2}{\rho_0^2}\,, \tag{14}$$

   and one on the r.h.s.:

$$-\frac{1}{\rho}\vec{\nabla}p = -\frac{1}{(\rho_0+\delta\rho)}\vec{\nabla}p\,, \tag{15}$$

   which when expanded in $\delta\rho$, δp gives us the linear term we already kept in eq. (??) and a quadratic piece of the form

$$\frac{1}{\rho_0}\frac{\delta\rho}{\rho_0}\vec{\nabla}p \sim k\,c_s^2\,\frac{\varepsilon^2}{\rho_0^2}\,, \tag{16}$$

   together with infinitely many higher order terms.

   In all cases, we see that the size of the non-linear terms neglected by the linear approximation, relative to the linear terms is of order ε/ρ_0. Therefore, the linear approximation for sound waves holds as long as

$$\varepsilon \ll \rho_0\,, \tag{17}$$

   whereas it breaks down for

$$\varepsilon \sim \rho_0\,. \tag{18}$$

   On physical grounds, this is very reasonable: it means that density perturbations can be consistently treated as small as long as their typical amplitude is smaller that the average density.

Sec 6 GEN II
Problem #6
Heinz

Solution:

a) Since the block of ice is very large, running the heat engine will just melt part of it to produce liquid water at 0 ^{0}C. The process will continue until the water of the heat source is cooled to 0 ^{0}C. At this point, the system will be in thermal equilibrium and no further work can be extracted.

b) The efficiency of an ideal heat engine running between the water at temperature T and the ice at temperature T_0 is

$$e(T) = 1 - T_0 / T$$

From the definition of efficiency, we have for an infinitesimal process at temp T:

$$e(T) = dW/dQ_H = (dQ_H - dQ_C)/ dQ_H = 1 - dQ_C/ dQ_H = 1 + dQ_C/ (mc\ dT).$$

Here mc is the heat capacity of the water; a minus sign has been introduced since dQ_H is positive as the temperature decreases. Equating the two expressions, we have

$$dQ_C = - mcT_0\, dT/ T$$

Integrating from $T = 373$ K to $T = 273$ K:

$$Q_C = mc\ T_0 \ln (373/273) = \underline{358 \text{ kJ.}}$$

c) The amount of melted ice will be

$$M = Q_C /L = \underline{1.08 \text{ kg}}$$

d) The amount of work done by the engine is

$$W = Q_H - Q_C = |\Delta T|\ mc - Q_C = 420 \text{ kJ} - 358 \text{ kJ} = 62 \text{ kJ}$$

e) The overall efficiency of conversion of heat to work is

$$E = W/ Q_H = 62 \text{ kJ}/420\text{kJ} = \underline{14.8\%}$$

5. A perfect fluid is described by the continuity and Euler equations, which govern the time-evolution of the density- and velocity- fields $\rho(\vec{x},t)$, $\vec{v}(\vec{x},t)$:

$$\dot{\rho} + \vec{\nabla} \cdot (\rho \vec{v}) = 0 \tag{1}$$

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admit wave-like solutions, which describe sound waves. Sound waves are just small density perturbations

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that propagate at the speed

$$c_s = \sqrt{\left.\frac{dp}{d\rho}\right|_{\rho=\rho_0}} \tag{6}$$

(a) Without keeping track of the vector structure and of order one coefficients, estimate from the equations of motion (1, 2) the amplitude of the fluctuations in $\vec{v}$ and p corresponding to the sound wave (5). Express your results in terms of ε, k, and c_s.

(b) Estimate the regime of validity of the linear approximation, that is find a condition on the quantities ε, k, and c_s that tells you when you can trust the linear approximation. Do you find your result reasonable, on physical grounds?

6. Consider running an ideal heat engine (at the Carnot efficiency) operating between the following two objects: 1 kg of water at 100 °C and a very large block of ice at 0 °C. We run the engine between this heat source and sink until work can no longer be extracted from the system.

In analyzing this problem, assume that the water and ice are thermally isolated from the external environment and neglect any effect of the changes in volume. Take the specific heat of water to be $c = 4.2$ kJ/kg-K, independent of temperature; the latent heat to melt ice is $L = 330$ kJ/kg.

At the completion of the process:

(a) What is the temperature of the water?

(b) How much heat has been absorbed by the block of ice during the process?

(c) How much ice has been melted?

(d) How much work has been done by the engine?

(e) What is the overall efficiency of the heat engine for the entire process?