Columbia University
Department of Physics
QUALIFYING EXAMINATION
Friday, January 16, 2009
1:00 PM - 3:00 PM

General Physics (Part I)
Section 5.

Two hours are permitted for the completion of this section of the examination. Choose 4 problems out of the 6 included in this section. (You will not earn extra credit by doing an additional problem). Apportion your time carefully.

Use separate answer booklet(s) for each question. Clearly mark on the answer booklet(s) which question you are answering (e.g., Section 5 (General Physics), Question 2; Section 5 (General Physics), Question 6, etc.)

Do **NOT** write your name on your answer booklets. Instead clearly indicate your **Exam Letter Code**.

You may refer to the single handwritten note sheet on 8 ½ x 11" paper (double-sided) you have prepared on General Physics. The note sheet cannot leave the exam room once the exam has begun. This note sheet must be handed in at the end of today's exam. Please include your Exam Letter Code on your note sheet. No other extraneous papers or books are permitted.

Simple calculators are permitted. However, the use of calculators for storing and/or recovering formulae or constants is NOT permitted.

Questions should be directed to the proctor.

Good luck!!

1. A star of radius $R_\star = 10^{11}$ cm and mass $M_\star = 10^{33}$ g moves on a parabolic orbit toward a black hole of mass $M = 10^{40}$ g. The angular momentum of the orbit is $L = 10^{23}$ cm^2/s (per unit mass). The black hole can be approximately described by the gravitational potential

$$\Phi = \begin{cases} -\frac{GM}{r-r_g} & r > r_g \\ -\infty & r < r_g \end{cases}$$

where $r_g = 2GM/c^2$, $c = 3\times10^{10}$ cm/s, and $G = 6.67\times10^{-8}\ \text{cm}^3/\text{s}^2\text{g}$ is the gravitational constant.

Does the star's orbit have a pericenter (point of minimum radius r)? Will the star survive the interaction with the black hole?

2. In condensed matter some vibration properties are represented by those of a linear chain of identical atoms of mass M that are joined by springs of force constant K (this is a simplest form of the harmonic approximation). The equilibrium length of each spring is a. The vibration modes are represented by plane waves that have propagation vector $\vec{k}$ and frequency ω. Here $|\vec{k}| = k = 2\pi/\lambda$, where λ is the mode wavelength. Assume that the chain has N atoms. Consider the 'longitudinal' waves that have displacements along the direction of the chain.

(a) Write the classical equation of motion for displacements u along the chain.

(b) Obtain the mode dispersion $\omega(\vec{k})$. Why are modes with k and $-k$ degenerate?

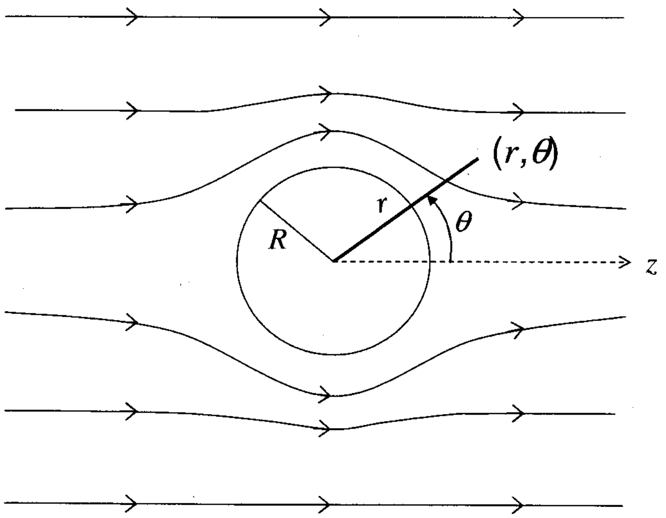
(c) Use cyclic boundary conditions to obtain the density of states of modes. How many modes are in the first Brillouin zone of the linear chain? Explain the result.

(d) Consider the long wavelength limit of $k \to 0$ and $\lambda/a \to \infty$. Show that the equation of motion for the displacements obtained in (a) reduces to a continuum elastic wave equation $\partial^2 u/dt^2 = v^2(\partial^2 u/dx^2)$, where x is the position along the direction of the chain. What is the speed v of the waves?

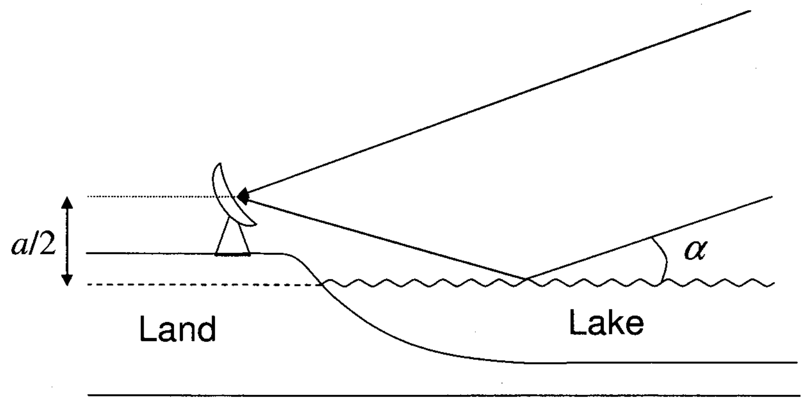
(e) Describe the Debye model for thermal properties due to vibration modes.

(f) Obtain the expression for the thermal energy due to vibrations in the chain within the Debye approximation. What is the low temperature limit of the specific heat due to these vibrations?

3. The steady-state flow of a fluid is specified by the fluid velocity $\vec{v}(\vec{r})$ at each position $\vec{r}$. Consider an incompressible fluid ($\nabla \cdot \vec{v} = 0$) undergoing irrotational flow ($\nabla\times\vec{v} = 0$). An impenetrable solid sphere of radius R is fixed in position while the fluid flows around it. Far from the sphere, the fluid flows uniformly in the z-direction ($\vec{v} = v_0\hat{z}$ for $r \to \infty$). For this azimuthally-symmetric problem, find the $\hat{r}$ and $\hat{\theta}$ components of $\vec{v}(r, \theta)$ for all $r > R$.



4. Imagine an antenna at the edge of a lake picking up a signal from a distant radio star (see figure below), which is just coming up above the horizon. Write an expression for the phase difference, δ, and for the angular position of the star when the antenna detects its first maximum. Express answers in terms of α and a.



5. (a) A box is filled with Planckian radiation of temperature $kT \gg m_e c^2$, where m_e is electron mass. A population of $e^{\pm}$ pairs is maintained in equilibrium with radiation through the reaction $e^+ + e^- \leftrightarrow \gamma + \gamma$. Find the number density of positrons in the box.

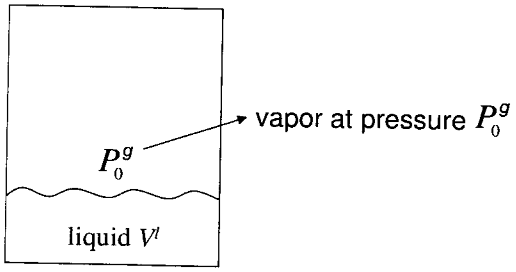
(b) Consider the same problem but now assume that the box is also filled with neutral electron-proton matter at the equilibrium temperature T. The proton number density n_p is given. Find the electron chemical potential μ assuming $\mu \ll kT$.

You can use the following integrals:

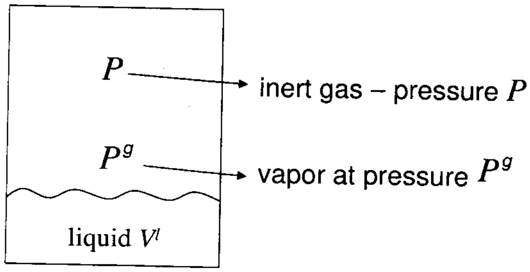
$$I_n = \int_0^\infty \frac{x^n dx}{e^x + 1} = \left(1 - \frac{1}{2^n}\right) n! \zeta(n+1)$$

where $\zeta(2) = \pi^2/6$, $\zeta(3) \approx 1.20$.

6. Consider a liquid and its vapor – both at the same temperature – in equilibrium with each other. Let the equilibrium vapor pressure be P_0^g. An inert, insoluble gas is added to the (closed) container holding the liquid. Find an expression for the new equilibrium vapor pressure of the liquid in terms of V^l, the volume of the liquid, P, the pressure of the inert gas, P_0^g, the vapor pressure without the inert gas, and T, the temperature of the system. Assume the gas phase of the liquid obeys the ideal gas law, and that the liquid is incompressible, i.e. $V^l(P) \approx V^l(P_0^g)$.



Before



After

Beloborodov . Astro

SOLUTIONS
Section 5
General Part I

Sec # 5 GEN I
Problem # 1
Beloborodov

GENERAL:

A star of radius $R_\star = 10^{11}$ cm and mass $M_\star = 10^{33}$ g moves on a parabolic orbit toward a black hole of mass $M = 10^{40}$ g. The angular momentum of the orbit is $L = 10^{23}$ cm^2/s (per unit mass). The black hole can be approximately described by the gravitational potential

$$\Phi = \begin{cases} -\dfrac{GM}{r - r_g} & r > r_g \\ -\infty & r < r_g \end{cases}$$

where $r_g = 2GM/c^2$, $c = 3 \times 10^{10}$ cm/s, and $G = 6.67 \times 10^{-8}$ cm^3/s^{2}g is the gravitational constant. Does the star's orbit have a pericenter (point of minimum radius r)? Will the star survive the interaction with the black hole?

Solution:

The star's velocity at the pericenter is $v = L/r$, if it exists. Then the pericenter radius is found from

$$\frac{L^2}{2r^2} - \frac{GM}{r - r_g} = 0 \quad \text{(parabolic orbit)}.$$

Denote $x = r/r_g$ and $l = L/cr_g$; then the equation gives

$$x^2 - l^2 x + l^2 = 0 \qquad \Rightarrow \qquad x = \frac{2}{1 - \sqrt{1 - 4/l^2}}.$$

(The root is chosen so that the Keplerian limit $l \gg 1$ gives the correct pericenter radius $r = L^2/2GM$.) The orbit has a pericenter, i.e. does not plunge into the black hole, if $l > 2$. In our case, $l \approx 2.2 > 2$. The pericenter radius is $r \approx 3.5 r_g$; $r_g \approx 1.5 \times 10^{12}$ cm.

The star is on an escaping orbit but it will not survive, because it will be disrupted by tidal forces. The star is stretched in the radial direction with tidal acceleration

$$a_T = \frac{\partial g}{\partial r} R_\star = \frac{2GMR_\star}{(r - r_g)^3} \qquad \text{where} \quad g = -\frac{\partial \Phi}{\partial r} = -\frac{GM}{(r - r_g)^2}.$$

Near the pericenter, this acceleration is much larger than $a_\star = GM_\star/R_\star^2$,

$$\frac{a_T}{a_\star} = \frac{M}{M_\star}\left(\frac{R_\star}{r - r_g}\right)^3 \sim 10^3.$$

This tidal acceleration is created for time $\Delta t \sim r/v$ where v is the star velocity near the pericenter, $v^2/2 = GM/(r - r_g)$. The velocity imparted by the tidal forces, $v_T \sim a_T r/v$, is larger than the escape velocity from the star, $v_\star = (2GM_\star/R_\star)^{1/2}$,

$$\frac{v_T^2}{v_\star^2} \sim \frac{a_T^2 r^2 R_\star}{2GM_\star v^2} = \frac{1}{4}\frac{M}{M_\star}\frac{R_\star^3 r^2}{(r - r_g)^5} = \frac{1}{4}\frac{a_T}{a_\star}\left(\frac{r}{r - r_g}\right)^2 \gg 1.$$

(a) We have a linear chain of period a

The positions of the atoms are

$$X_n = n a \qquad n = 0, 1, 2 \ldots$$

The displacements are

$$u(x_n) = u(na)$$

The equation of motion is derived from the condition

$$M \ddot{u}(x_n) = - \frac{\partial V}{\partial u(x_n)}$$

$$V = \frac{1}{2} K \sum_n \left[u(x_n) - u(x_{n+1}) \right]^2$$

The equation of motion is

$$M \ddot{u}(x_n) = -K \left[2u(x_n) - u(x_{n-1}) - u(x_{n+1}) \right]$$

Look for solutions that are.

$$u(x_n) = A e^{i(k x_n - \omega t)}$$

(b) $$\omega(k) = \sqrt{\frac{2K}{M}(1-\cos ka)}$$

$$= 2\sqrt{\frac{K}{M}}\,\left|\sin\tfrac{1}{2}ka\right|$$

the modes with k and $-k$ are degenerate because the direction of propagation does not change the mode frequency.

(c) Cyclic boundary conditions mean

$$u(x_n) = u(x_n + Na)$$

This requires that

$$k = \frac{2\pi}{a}\,\frac{l}{N} \qquad l = 0, 1, 2 \ldots$$

The first Brillouin ~~zone~~ has

$$-\pi/a \le k \le \pi/a$$

It has all the N degrees of freedom for one orientation of the displacement $u(x_n)$.

The density of states in k-space is

$$g(k) = \frac{Na}{2\pi}$$

In frequency space it is

$$g(\omega) = \frac{g(k)}{(d\omega/dk)}$$

The number of modes in the 1st Brillouin zone is

$$\int_{-\pi/a}^{\pi/a} g(k)\, dk = N$$

(d) Write the right handside of the eq. in (a)

$$-K\left[2u(x_n) - u(x_{n-1}) - u(x_{n+1})\right]$$

$$= K\left[\left(u(x_{n+1}) - u(x_n)\right) - \left(u(x_n) - u(x_{n-1})\right)\right]$$

In the limit $\lambda \gg a$

$$u(x_{n+1}) - u(x_n) = \frac{\partial u(x_{n+1})}{\partial x_{n+1}}\, a$$

and

$$u(x_n) - u(x_n) = \frac{\partial u(x_n)}{\partial x_n} a$$

so that the right hand side of the eq. in (a) is

$$Ka \left[\frac{\partial u(x_{n+1})}{\partial x_{n+1}} - \frac{\partial u(x_n)}{\partial x_n} \right]$$

$$= Ka \left[\frac{\partial^2 u}{\partial x^2} a \right] = Ka^2 \frac{\partial^2 u}{\partial x^2}$$

The value of the velocity is

$$v = \sqrt{\frac{K}{M}}\, a$$

(e) The Debye model assumes

(i) A dispersion $\omega(k) = vk$

(ii) A cut-off wave vector k_D

given by the condition

$$\int_{-k_D}^{k_D} g(k)\, dk = N$$

(f) The thermal energy is

$$E_D \sim (k_B T)^2 \int_0^\infty \frac{x}{e^x - 1}$$

The specific heat is

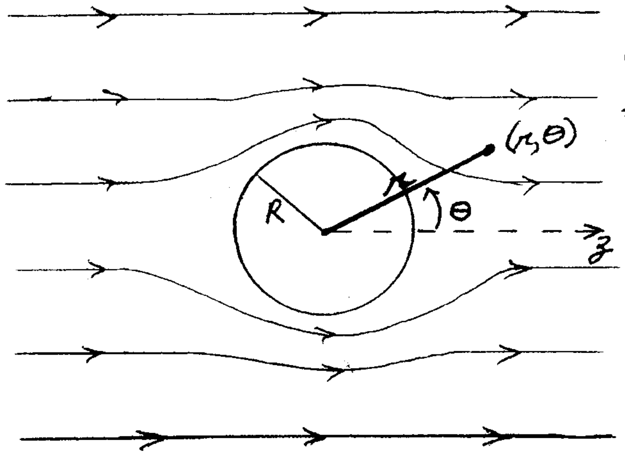
$$C_V \sim T$$

Blaer · General Fluids

Sec 5 GEN I
Problem # 3
Blaer

Qualifying Examination Problem
Fluids

Allan Blaer
December 2008



The steady-state flow of a fluid is specified by the fluid velocity $\vec{v}(\vec{r})$ at each position $\vec{r}$. Consider an incompressible fluid ($\text{div}\,\vec{v} = 0$) undergoing irrotational flow ($\text{curl}\,\vec{v} = 0$). An impenetrable solid sphere of radius R is fixed in position while the fluid flows around it. Far from the sphere, the fluid flows uniformly in the z-direction ($\vec{v} = v_0\hat{z}$ for $r \longrightarrow \infty$). For this azimuthally-symmetric problem, find the $\hat{r}$ and $\hat{\theta}$ components of $\vec{v}(r,\theta)$ for all r>R.

<u>Solution</u>: $\text{curl}\,\vec{v} = 0 \Rightarrow \vec{v} = -\nabla\Phi$. $\text{div}\,\vec{v} = 0 \Rightarrow \nabla^2\Phi = 0$.

For azimuthally-symmetric solutions to Laplace's equation,

$$\Phi(r,\theta) = \sum_{l=0}^{\infty} \frac{A_l}{r^{l+1}} P_l(\cos\theta) + \sum_{l=0}^{\infty} B_l r^l P_l(\cos\theta).$$

$$r \to \infty \Rightarrow \vec{v} = v_0\hat{z} \Rightarrow \Phi = -v_0 z = -v_0 r\cos\theta = -v_0 r P_1(\cos\theta).$$

$$\therefore B_{l=1} = -v_0 \text{ and } B_{l>1} = 0.$$

$$\therefore \Phi(r,\theta) = -v_0 r P_1(\cos\theta) + B_0 + \sum_{l=0}^{\infty} \frac{A_l}{r^{l+1}} P_l(\cos\theta)$$

At $r=R$, $v_r = 0$ (impenetrable sphere). $\therefore \left(\frac{\partial\Phi}{\partial r}\right)_{r=R} = 0.$

$$\therefore -v_0 P_1(\cos\theta) + \sum_{l=0}^{\infty} \frac{-A_l(l+1)}{R^{l+2}} P_l(\cos\theta) = 0 \text{ for all } \theta.$$

$$\therefore A_{l=1} = \frac{-v_0 R^3}{2} \text{ and } A_{l\neq 1} = 0.$$

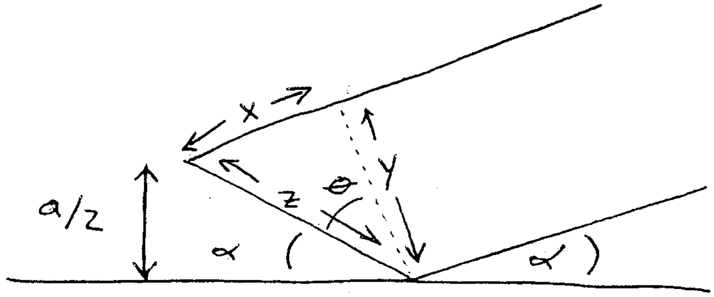
$$\therefore \Phi(r,\theta) = B_0 - v_0\left(r + \frac{R^3}{2r^2}\right)\underbrace{P_1(\cos\theta)}_{\cos\theta}$$

$$v_r = \frac{-\partial\Phi}{\partial r} = +v_0\left(1 - \frac{R^3}{r^3}\right)\cos\theta$$

$$v_\theta = -\frac{1}{r}\frac{\partial\Phi}{\partial\theta} = -v_0\left(1 + \frac{R^3}{2r^3}\right)\sin\theta$$

OPTICS

Sec 5 GEN I
Problem # 4
Hughes



$\delta = kz - kx + \pi$ ← water reflection

$$\sin\alpha = \frac{a/2}{z} \Rightarrow z = \frac{a}{2\sin\alpha}$$

$$\theta = 90 - 2\alpha \qquad \sin\theta = \frac{x}{z}$$

$$x = z\sin\theta = \frac{a}{2\sin\alpha}\sin(90-2\alpha)$$

$$= \frac{a\cos 2\alpha}{2\sin\alpha}$$

$$\boxed{\delta = \frac{ka}{2\sin\alpha}(1-\cos 2\alpha) + \pi}$$

Maximum occurs @ $\delta = 2\pi$

$$\pi = \frac{ka}{\sin\alpha}\left(\frac{1-\cos 2\alpha}{2}\right) = \frac{ka}{\sin\alpha}\sin^2\alpha = ka\sin\alpha = \frac{2\pi a\sin\alpha}{\lambda}$$

First maximum @ $\boxed{\alpha = \sin^{-1}\left(\frac{\lambda}{2a}\right)}$

Beloborodov Stat · Mech

Sec 5 GEN I
Problem # 5
Beloborodov

MODERN:

(a) A box is filled with Planckian radiation of temperature $kT \gg m_e c^2$, where m_e is electron mass. A population of $e^{\pm}$ pairs is maintained in equilibrium with radiation through reaction $e^+ + e^- \leftrightarrow \gamma + \gamma$. Find the number density of positrons in the box.

(b) Consider the same problem but now assume that the box is also filled with neutral electron-proton matter at the equilibrium temperature T. The proton number density n_p is given. Find the electron chemical potential μ assuming $\mu \ll kT$.

You can use the following integrals:

$$I_n = \int_0^\infty \frac{x^n\, dx}{e^x + 1} = \left(1 - \frac{1}{2^n}\right) n!\, \zeta(n+1),$$

where $\zeta(2) = \pi^2/6$, $\zeta(3) \approx 1.20$.

Solution:

(a) The $e^{\pm}$ are relativistic ($kT \gg m_e c^2$) and their energies are $E = cp$ where p is the particle momentum. The occupation number of $e^{\pm}$ is described by Fermi-Dirac function

$$f = \frac{1}{\exp\left(\dfrac{cp - \mu_\pm}{kT}\right) + 1},$$

where μ_+ and μ_- are the chemical potentials of e^+ and e^-. The number densities of e^+ and e^- are

$$n_\pm = \frac{2}{h^3} \int_0^\infty f\, 4\pi p^2 dp. \tag{1}$$

Since $e^{\pm}$ are in equilibrium with photons (which have zero chemical potential), $\mu_+ + \mu_- = 0$. By symmetry, $\mu_+ = \mu_-$ and hence $\mu_+ = \mu_- = 0$. Then equation (1) gives

$$n_+ = n_- = \frac{3\zeta(3)\,\theta^3}{2\pi^2 \lambda\!\!\!^{-3}}, \qquad \text{where} \quad \theta = \frac{kT}{m_e c^2}, \qquad \lambda\!\!\!^- = \frac{\hbar}{m_e c}.$$

(b) Electrons and positrons are still in equilibrium with radiation, therefore $\mu_+ = -\mu_- \equiv \mu$. Neutrality requires $n_- - n_+ = n_p \;\Rightarrow\; n_+ \neq n_- \;\Rightarrow\; \mu \neq 0$. Using $\mu \ll kT$, one can expand the integral in equation (1) in μ/kT, keeping the linear term. Then one finds

$$n_\pm = \frac{1}{\pi^2 \lambda\!\!\!^{-3}} \left[\frac{3}{2}\zeta(3)\,\theta^3 \mp \frac{\pi^2}{6}\,\theta^2\,\frac{\mu}{kT}\right],$$

$$n_- - n_+ = n_p \qquad \Rightarrow \qquad \frac{\mu}{kT} = \frac{3\lambda\!\!\!^{-3} n_p}{\theta^2}.$$

Sec 5 GEN I
Problem # 6
Hailey

Solution thermo I

The liquid and gas phases are in equilibrium so The Gibb's free energies are equal

And for $dG = -SdT + VdP + \mu dn$

we have for $dT = 0$ $dn = 0$

$$V^l dP^l = V^g dP^g \quad \text{And} \quad P^g V^g = RT$$

$$V^l dP^l = V^g dP^g = RT \, dP^g / P^g$$

$$\int_0^P \frac{V^l}{RT} dP^l = \int_{P_0^g}^{P^g(P)} dP^g / P^g$$

$$\frac{V^l}{RT} P = \ln \frac{P^g(P)}{P_0^g}$$

$$P^g(P) = P_0^g \, e^{\frac{V^l P}{RT}}$$

Ans