# Columbia University
# Department of Physics
# QUALIFYING EXAMINATION
# Wednesday, January 14, 2009
# 3:10 PM - 5:10 PM

## Modern Physics
## Section 4. Relativity and Applied Quantum Mechanics

Two hours are permitted for the completion of this section of the examination. Choose 4 problems out of the 5 included in this section. (You will not earn extra credit by doing an additional problem). Apportion your time carefully.

Use separate answer booklet(s) for each question. Clearly mark on the answer booklet(s) which question you are answering (e.g., Section 4 (Relativity and Applied QM), Question 2, etc.)

Do **NOT** write your name on your answer booklets. Instead clearly indicate your **Exam Letter Code**.

You may refer to the single handwritten note sheet on 8 ½ x 11" paper (double-sided) you have prepared on Modern Physics. The note sheet cannot leave the exam room once the exam has begun. This note sheet must be handed in at the end of today's exam. Please include your Exam Letter Code on your note sheet. No other extraneous papers or books are permitted.

Simple calculators are permitted. However, the use of calculators for storing and/or recovering formulae or constants is NOT permitted.

Questions should be directed to the proctor.

Good luck!!

**1.** Four electrons are localized in the four vertices of a tetrahedron. Due to the strong Coulomb repulsion the transition of the electrons between the vertices is forbidden, so the only low energy degrees of freedom are the electron spins, $\hat{\vec{s}}^{(i)}, i = 1, 2, 3, 4;\ s^{(i)} = \frac{1}{2}$.

(a) What is the total number of states?

(b) Let the system be described by the exchange Hamiltonian

$$\widehat{H} = K \sum_{i>j} \hat{\vec{s}}^{(i)} \hat{\vec{s}}^{(j)} \tag{1}$$

Find the energy levels and their degeneracies.

(c) The exchange Hamiltonian (1) can be generalized to include the interactions of more than two spins. How many of those higher order interactions are allowed by spatial and time reversal symmetries? (Neglect the spin-orbit interaction)

(d) A generalization of Eq. (1) may have the form

$$\begin{aligned}\widehat{H} &= K \sum_{i>j} \hat{\vec{s}}^{(i)} \hat{\vec{s}}^{(j)} \\ &+ K' \left(\hat{\vec{J}}^2\right)^2 \left[\left(\hat{\vec{s}}^{(1)} \cdot \hat{\vec{s}}^{(2)}\right)\left(\hat{\vec{s}}^{(3)} \cdot \hat{\vec{s}}^{(4)}\right) + \left(\hat{\vec{s}}^{(1)} \cdot \hat{\vec{s}}^{(3)}\right)\left(\hat{\vec{s}}^{(2)} \cdot \hat{\vec{s}}^{(4)}\right) \right. \\ &+ \left. \left(\hat{\vec{s}}^{(1)} \cdot \hat{\vec{s}}^{(4)}\right)\left(\hat{\vec{s}}^{(2)} \cdot \hat{\vec{s}}^{(3)}\right)\right]\end{aligned} \tag{2}$$

Find the energy levels and their degeneracies.

**2.** If an atom or nucleus makes a transition from state A to state B emitting a photon of energy $E$ equal to the energy difference between the two states, an identical atom or nucleus in state B can absorb that photon and go to state A. This process, called *resonance fluorescence*, will not proceed, however, if recoil by the emitter (or absorber) reduces the photon energy by an amount greater than the line-width of the transition.

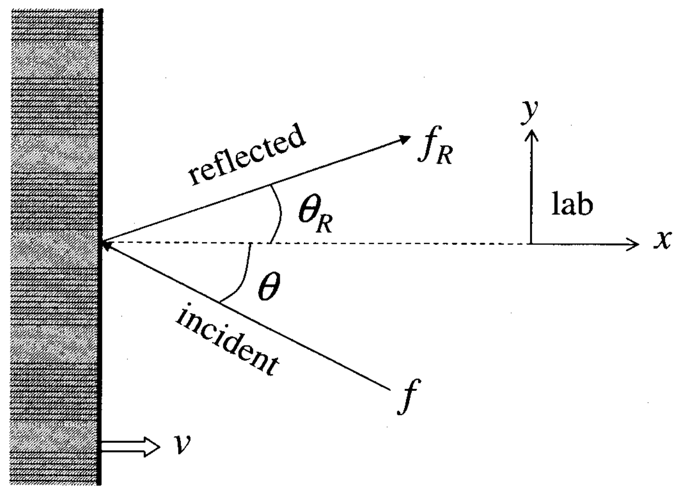
(a) Using typical lifetimes of atomic ($10^{-8}$ s) and nuclear ($10^{-10}$ s) transitions, estimate the typical line-widths for such transitions.

(b) Using typical photon energies in atomic and nuclear transitions show that recoil will not stop atomic resonance fluorescence but will stop nuclear resonance fluorescence.

(c) In 1958 Rudolf Mössbauer discovered the effect that bears his name and for which he won the 1961 Nobel prize. This effect can drastically reduce the recoil energy loss in a nuclear gamma transition. How does it work?

(d) An early application of this effect was made by Robert Pound and Glen Rebka who allowed photons from the nuclear gamma transition of $^{57}Co$ to fall down the Harvard Tower. Assuming that the transition is a typical nuclear gamma transition and that the tower is a typical tower, estimate the expected energy shift and argue that the Mössbauer effect allows one to detect such a shift. What is so important about this experiment?

**3.** A monochromatic beam of light is incident on a flat mirror. With respect to the laboratory, the mirror is traveling in at relativistic speed $v$ in the $+x$ direction. (The plane of the mirror is perpendicular to the $x$-axis.) Also with respect to the laboratory, the incident light beam has frequency $f$ and is traveling at angle $\theta$ with respect to the $x$-axis. Find the frequency $f_R$ and the angle $\theta_R$ of the reflected light beam as measured in the laboratory.



4. A capacitor is formed from two large parallel conducting plates of proper area $A_p$. As seen in a laboratory frame, one plate is moving to the left and the other is moving to the right, both with velocity (magnitude) $v$. Ignore fringe field effects in answering the following questions. Carry out the following calculations for the situation when the two plates are fully overlapped.

(a) What is the ratio ($R$) of the capacitance of the moving plate capacitor to a capacitor with the same geometry but with static plates (evaluate in laboratory frame)?

(b) If the potential difference between the plates is $V$, what is the pressure due to electromagnetic forces on one of the plates as seen in the rest frame of the plate?

(c) If the potential difference between the plates is $V$, what is the pressure due to electromagnetic forces on one of the plates as seen in the laboratory frame?

**5.** A photon of wavelength $\lambda$ Compton scatters off a free electron (mass $m$) which is initially at rest. The scattered photon has wavelength $\lambda'$ and scatters at an angle $\theta$ (measured from the incident direction). Express your answers in terms of $m$, $\lambda$, $\theta$ and $h$.

(a) Calculate $\lambda'$.

(b) Calculate the kinetic energy of the recoil electron.

Section # 4 CEL
Problem # 1
Aleiner

**Solution**

1. $2^4 = 16$.

2. Introduce the operator of total momentum:

$$\hat{\vec{J}} = \hat{\vec{s}}^{(1)} + \hat{\vec{s}}^{(2)} + \hat{\vec{s}}^{(3)} + \hat{\vec{s}}^{(4)}. \quad (3)$$

Obviously

$$\hat{\vec{J}}^2 = 3 + 2\sum_{i<j} \hat{\vec{s}}^{(i)} \cdot \hat{\vec{s}}^{(j)}. \quad (4)$$

Thus, the Hamiltonian (1) can be rewritten as

$$\hat{H} = \frac{K}{2}\left(\hat{\vec{J}}^2 - 3\right). \quad (5)$$

Eigenvalues of Eq. (5) are classified according to their total angular momenta $\hat{\vec{J}}^2 = J(J+1)$:

$$\begin{aligned} J = 0;&\quad E = -\frac{3K}{2};\quad 2\times 1 - \text{folded};\\ J = 1;&\quad E = -\frac{K}{2};\quad 3\times 3 - \text{folded};\\ J = 2;&\quad E = \frac{3K}{2};\quad 5 - \text{folded}. \end{aligned} \quad (6)$$

3. Terms with odd number of spins are forbidden by the time of the time reversal symmetry. Thus only terms involving either two or four spins are allowed. As all the vertices in tetrahedron are equivalent the only paiwise interaction allowed is given by Eq. (1).

Furthermore,

$$\hat{s}_\alpha^{(1)}\hat{s}_\beta^{(1)} = \frac{1}{4}\delta_{\alpha\beta} + \frac{i}{2}\epsilon^{\alpha\beta\gamma}\hat{s}_\gamma^{(1)};\ \alpha,\beta,\gamma = x,y,z. \quad (7)$$

It yields

$$\begin{aligned} &\left(\hat{\vec{s}}^{(1)}\cdot\hat{\vec{s}}^{(2)}\right)^2 = \frac{3}{16} - \frac{1}{2}(\hat{\vec{s}}^{(1)}\cdot\hat{\vec{s}}^{(2)};\\ &\left(\hat{\vec{s}}^{(1)}\cdot\hat{\vec{s}}^{(2)}\right)\left(\hat{\vec{s}}^{(1)}\cdot\hat{\vec{s}}^{(3)}\right) + \left(\hat{\vec{s}}^{(1)}\cdot\hat{\vec{s}}^{(3)}\right)\left(\hat{\vec{s}}^{(1)}\cdot\hat{\vec{s}}^{(2)}\right) = \frac{1}{2}(\hat{\vec{s}}^{(2)}\cdot\hat{\vec{s}}^{(3)}; \end{aligned} \quad (8)$$

This means that only combination where all the spins are different generate something new in the Hamiltonian. As all the vertices in tetrahedron are equivalent, Hamiltonian (2) is the only possibility without involving the spin-orbit interactions.

4. Consider

$$\left(\hat{\vec{J}}^2\right)^2 = \left(3 + 2\sum_{i<j} \hat{\vec{s}}^{(i)} \cdot \hat{\vec{s}}^{(j)}\right)^2. \tag{9}$$

Using Eq. (8), we find

$$\begin{aligned}
&\left(\hat{\vec{J}}^2\right)^2 = 9 + 12\sum_{i<j} \hat{\vec{s}}^{(i)} \cdot \hat{\vec{s}}^{(j)} \\
&+ 8\left[(\hat{\vec{s}}^{(1)} \cdot \hat{\vec{s}}^{(2)})(\hat{\vec{s}}^{(3)} \cdot \hat{\vec{s}}^{(4)}) + (\hat{\vec{s}}^{(1)} \cdot \hat{\vec{s}}^{(3)})(\hat{\vec{s}}^{(2)} \cdot \hat{\vec{s}}^{(4)}) + (\hat{\vec{s}}^{(1)} \cdot \hat{\vec{s}}^{(4)})(\hat{\vec{s}}^{(2)} \cdot \hat{\vec{s}}^{(3)})\right] \\
&+ 4\sum_{i<j}\left(\frac{3}{16} - \frac{1}{2}(\hat{\vec{s}}^{(i)} \cdot \hat{\vec{s}}^{(2)}\right) \\
&+ 4 * 2 * \frac{1}{2}\sum_{i<j} \hat{\vec{s}}^{(i)} \cdot \hat{\vec{s}}^{(2)} \\
&= \frac{27}{2} + 14\sum_{i<j} \hat{\vec{s}}^{(i)} \cdot \hat{\vec{s}}^{(2)} + 8\left[(\hat{\vec{s}}^{(1)} \cdot \hat{\vec{s}}^{(2)})(\hat{\vec{s}}^{(3)} \cdot \hat{\vec{s}}^{(4)}) + (\hat{\vec{s}}^{(1)} \cdot \hat{\vec{s}}^{(3)})(\hat{\vec{s}}^{(2)} \cdot \hat{\vec{s}}^{(4)}) + (\hat{\vec{s}}^{(1)} \cdot \hat{\vec{s}}^{(4)})(\hat{\vec{s}}^{(2)} \cdot \hat{\vec{s}}^{(3)})\right] \\
&= -\frac{15}{2} + 7\hat{\vec{J}}^2 + 8\left[(\hat{\vec{s}}^{(1)} \cdot \hat{\vec{s}}^{(2)})(\hat{\vec{s}}^{(3)} \cdot \hat{\vec{s}}^{(4)}) + (\hat{\vec{s}}^{(1)} \cdot \hat{\vec{s}}^{(3)})(\hat{\vec{s}}^{(2)} \cdot \hat{\vec{s}}^{(4)}) + (\hat{\vec{s}}^{(1)} \cdot \hat{\vec{s}}^{(4)})(\hat{\vec{s}}^{(2)} \cdot \hat{\vec{s}}^{(3)})\right]
\end{aligned} \tag{10}$$

Therefore, Hamiltonian (2) can be rewritten as

$$\hat{H} = -\frac{3K}{2} + \frac{15K'}{16} + \left(\frac{K}{2} - \frac{7K'}{8}\right)\hat{\vec{J}}^2 + \frac{K'}{8}(\hat{\vec{J}}^2)^2; \tag{11}$$

Eigenvalues of Eq. (11) are classified according to their total angular momenta $\hat{\vec{J}}^2 = J(J+1)$:

$$\begin{aligned}
J = 0;&\quad E = -\frac{3K}{2} + \frac{15K'}{16};\quad 2 \times 1 - \text{folded};\\
J = 1;&\quad E = -\frac{K}{2} - \frac{5K'}{16};\quad 3 \times 3 - \text{folded};\\
J = 2;&\quad E = \frac{3K}{2} + \frac{3K'}{16};\quad 5 - \text{folded}.
\end{aligned} \tag{12}$$

Ponton solution
Appl QM-RFL
SEC 4 Problem # 2

Soln. Mössbauer

Will need approximate conversions, e.g.

$$1\ \text{fm} = 10^{-15}\text{m} \leftrightarrow (200\ \text{MeV})^{-1} \quad \text{(nuclear scales)}$$

$$1\ \text{m} \leftrightarrow (2\times10^{-13}\ \text{MeV})^{-1}$$

$$\frac{1\text{m}}{c} = \tfrac{1}{3}\times10^{-8}\ \text{s} \leftrightarrow (2\times10^{-13}\ \text{MeV})^{-1}$$

$$1\ \text{s} \leftrightarrow (10^{-21}\ \text{MeV})^{-1}$$

a) Atomic: $\Gamma_A \sim \tau^{-1} \sim 10^{8}\ \text{s}^{-1} \sim 10^{-13}\ \text{MeV}$

Nuclear: $\Gamma_N \sim 10^{10}\ \text{s} \sim 10^{-11}\ \text{MeV}$

b) Nuclear recoil energy $E_R = \frac{p^2}{2M_N} = \frac{E_\gamma^2}{2M_N}$. Use $M_N \sim 50$ GeV.

Typical atomic photon energy $E_\gamma \sim \text{eV} \Rightarrow E_R^A \sim \frac{(10^{-6}\ \text{MeV})^2}{100\times10^3\ \text{MeV}} = 10^{-17}\ \text{MeV}$

" nuclear " " $E_\gamma \sim \text{MeV} \Rightarrow E_R^N \sim 10^{-5}\ \text{MeV}$

(could be 10-100 keV $\Rightarrow E_R \sim 10^{-3} - 10^{-1}$ eV)

Hence $E_R^A \ll \Gamma_A$ but $E_R^N \gg \Gamma_N$

$\hookrightarrow \gamma$ non-resonant

c) Mössbauer: nucleus in a lattice $\rightarrow$ whole solid can absorb momentum with negligible $E_R$ since huge recoiling mass

(there is some probability that no phonons are excited, so emission is essentially recoil-free in such cases)

d) Pound & Rebka: measure gravitational (blue) shift for falling photon.

Estimate using effective photon mass: $m_\gamma = \frac{E_\gamma}{c^2}$

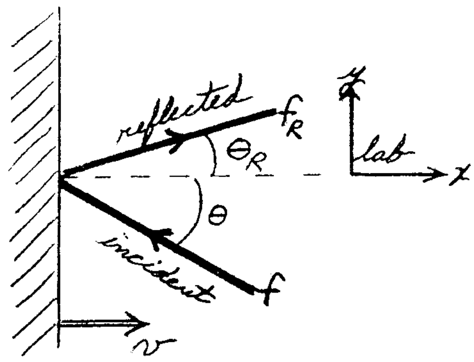
$$\Rightarrow \Delta E_\gamma \sim m_\gamma g h \Rightarrow \frac{\Delta E_\gamma}{E_\gamma} \sim \frac{gh}{c^2} \sim \frac{10\cdot10}{(3\times10^8)^2} \sim 10^{-15} \quad (\text{for } h = 10\text{m})$$

For $E_\gamma \sim 1\ \text{MeV} \Rightarrow \Delta E_\gamma \sim 10^{-15}\ \text{MeV}$ (without Mössbauer effect, no hope to see this)

Qualifying Examination
Relativity Problem

December 2008
Allan Blaer - Relativity

Problem # 3




A monochromatic beam of light is incident on a flat mirror. With respect to the laboratory, the mirror is traveling at relativistic speed v in the +x direction. (The plane of the mirror is perpendicular to the x-axis.) Also with respect to the laboratory, the incident light beam has frequency f and is traveling at angle $\theta$ with respect to the x-axis. Find the frequency $f_R$ and the angle $\theta_R$ of the reflected light beam as measured in the laboratory.

**Solution**: Let $(\omega = 2\pi f)$ and use $(c \equiv 1)$. Then, $(\omega, \vec{k})$ transforms as a 4-vector under Lorentz transformations, with $|\vec{k}| = \omega$.
Let x'-y' axes be fixed with respect to the mirror (moving at $v_x = +v$ with respect to the lab axes x-y). $\left(\gamma \equiv \frac{1}{\sqrt{1-v^2}}\right)$

Incident beam (mirror frame): $\omega' = \gamma(\omega - v k_x)$ and $k_x' = \gamma(k_x - v\omega)$ and $k_y' = k_y$ as measured in the mirror frame of reference.

In the mirror frame, the incident & reflected beams have the same frequency and the angle of incidence equals the angle of reflection.

Reflected beam (mirror frame): $\omega_R' = \omega'$, $k_{Rx}' = -k_x'$, $k_{Ry}' = k_y'$.

We now Lorentz transform the reflected beam's $(\omega_R', \vec{k}_R')$ back to the laboratory frame (which moves at $v_x = -v$ with respect to the mirror):

$$\omega_R = \gamma(\underbrace{\omega_R'}_{\omega'} + v\underbrace{k_{Rx}'}_{-k_x'}) \text{ and } k_{Rx} = \gamma(\underbrace{k_{Rx}'}_{-k_x'} + v\underbrace{\omega_R'}_{\omega'}) \text{ and } k_{Ry} = \underbrace{k_{Ry}'}_{k_y'}.$$

$$\therefore \omega_R = \gamma(\omega' - v k_x') \text{ and } k_{Rx} = \gamma(-k_x' + v\omega') \text{ and } k_{Ry} = k_y'$$

where $\omega' = \gamma(\omega - v k_x)$, $k_x' = \gamma(k_x - v\omega)$, $k_y' = k_y$.

$$\therefore \omega_R = \gamma^2[\omega(1+v^2) - 2v\underbrace{k_x}_{(-\omega\cos\theta)}] \text{ and } \underbrace{k_{Rx}}_{(\omega_R\cos\theta_R)} = \gamma^2[2v\omega - (1+v^2)\underbrace{k_x}_{(-\omega\cos\theta)}] \text{ and } \underbrace{k_{Ry}}_{(\omega_R\sin\theta_R)} = \underbrace{k_y}_{(\omega\sin\theta)}.$$

($\omega_R = 2\pi f_R$, $\omega = 2\pi f$)

$$\therefore \boxed{\omega_R = \frac{\omega}{(1-v^2)}\left[1 + 2v\cos\theta + v^2\right]} \text{ and } \boxed{\tan\theta_R = \frac{(1-v^2)\sin\theta}{[2v + (1+v^2)\cos\theta]}}$$

Cole- Relativity
Sec 4 REL
Problem #4
Cole

Quals problem solutions: Moving plate capacitor

a) An applied potential difference will generate a surface charge density and that surface charge will generate a surface current density. The choice of directions for the plates means that the surface currents are in the same direction. Thus, the magnetic field between the surfaces of the conductor in the laboratory frame is zero but, the magnetic field outside the surface currents, and thus is the conductors is non-zero. Thus, the usual condition for zero electric field in the conductor will not apply. To apply the standard condition on fields at conductor surfaces, we can transform to the rest frame of one of the plates. The electric field in that frame is given by $E' = \gamma E$, since the is zero magnetic field between the surface charges/currents. So, the induced surface charge in the rest frame of one of the plates has magnitude $\sigma' = E'/4\pi = \gamma E/4\pi$. In that frame the surface current is zero, so we can easily transform the charge density back to the laboratory frame, $\sigma = \gamma\sigma' = \gamma^2 E/4\pi$. The total charge stored on the plate is $Q = \sigma A$ where the area is smaller than the proper area due to Lorentz contraction, $A = A_p/\gamma$. So, we have $Q = A_p\, \gamma E/4\pi = A_p\, \gamma V/4\pi d$. Thus, the ratio of the capacitance to that of static plates is $R = \gamma$.

If this part is analyzed in the laboratory frame, then the surface charge density is that which produces an electric field in the conductor that exactly cancels the magnetic forces. The magnetic field is $B = 4\pi\varkappa/c$ where $\varkappa = \sigma v$. So, $B = 4\pi\beta\sigma$. The magnetic force on a hypothetical charge moving with the plate is $F = q\beta B = q(4\pi\beta^2\sigma)$. The electric field in the conductor would be $E_c = E - 4\pi\sigma$. We want $qE_c + F = 0$ (signs are correct). Or we want $E - 4\pi\sigma = -4\pi\beta^2\sigma$ or

$$\sigma\left(1-\beta^2\right) = \frac{E}{4\pi} \Rightarrow \sigma = \gamma^2\frac{E}{4\pi}$$

consistent with the above result.

b) The pressure on the plate in the plate rest frame is simply $P' = \sigma' E'/2 = \gamma^2 E^2/8\pi$ or $P = \gamma^2 V^2/8\pi d^2$.

c) The simplest way to obtain the pressure in the laboratory frame is to transform the force from the conductor rest frame in part b. The total force on the plate in the plate rest frame is $F' = P' A_p$. The component of the force normal to the boost direction is reduced by a factor of $\gamma$ under a Lorentz boost so the total force in the laboratory frame is $F = P' A_p/\gamma$. The pressure in the laboratory frame is therefore, $P = F/A = P'$. So the pressure in the laboratory frame is the same as the result in part b.

Sec 4 REL
Problem # 5 Tuts

Relativity problem solution


From conservation of energy

$$pc + mc^2 = p'c + \sqrt{p_e^2c^2 + m^2c^4}$$

$$(pc + mc^2 - p'c)^2 = p_e^2c^2 + m^2c^4$$

and conservation of momentum

$$\vec{p} - \vec{p}' = \vec{p}_e \Rightarrow (\vec{p} - \vec{p}')^2 = p_e^2$$

$$\therefore p^2 + p'^2 + m^2c^2 - 2pp' - 2p'mc + 2pmc - p^2 - p'^2 + 2\vec{p}\cdot\vec{p}' = m^2c^2$$

$$pp' - \vec{p}\cdot\vec{p}' = (p - p')mc$$

$$pp'(1 - \cos\theta) = (p - p')mc$$

$$1 - \cos\theta = \left(\frac{1}{p'} - \frac{1}{p}\right)mc$$

where $p = \frac{h}{\lambda}$

$$\therefore \lambda' - \lambda = \frac{h}{mc}(1 - \cos\theta)$$

$$\boxed{\lambda' = \lambda + \frac{h}{mc}(1 - \cos\theta)}$$

b. from above $\frac{1}{p'c} = \frac{1}{pc} + \frac{1}{mc^2}(1 - \cos\theta)$

$$\therefore p'c = \frac{mc^2}{1 - \cos\theta + \frac{mc}{p}}$$

and $K = \sqrt{p_e^2c^2 + m^2c^4} - mc^2 = pc - p'c$

but from energy conservation that equals $pc - p'c$ (from 1st equation in a)

$$\therefore K = pc - \frac{mc^2}{1 - \cos\theta + \frac{mc}{p}}$$

$$K = \frac{pc(1 - \cos\theta + \frac{mc}{p}) - mc^2}{1 - \cos\theta + \frac{mc}{p}}$$

$$K = pc(1 - \cos\theta)$$

$$K = \frac{pc(1-\cos\theta)}{1-\cos\theta + \frac{mc}{p}}$$

$$\boxed{K = \frac{(1-\cos\theta)\frac{hc}{\lambda}}{1-\cos\theta + \frac{mc\lambda}{h}}}$$