Columbia University
Department of Physics
QUALIFYING EXAMINATION
Wednesday, January 14, 2009
1:00 PM - 3:00 PM

Modern Physics
Section 3. Quantum Mechanics

Two hours are permitted for the completion of this section of the examination. Choose 4 problems out of the 5 included in this section. (You will not earn extra credit by doing an additional problem). Apportion your time carefully.

Use separate answer booklet(s) for each question. Clearly mark on the answer booklet(s) which question you are answering (e.g., Section 3 (QM), Question 1; Section 3 (QM) Question 5, etc.)

Do **NOT** write your name on your answer booklets. Instead clearly indicate your **Exam Letter Code**.

You may refer to the single handwritten note sheet on 8 ½ x 11" paper (double-sided) you have prepared on Modern Physics. The note sheet cannot leave the exam room once the exam has begun. This note sheet must be handed in at the end of today's exam. Please include your Exam Letter Code on your note sheet. No other extraneous papers or books are permitted.

Simple calculators are permitted. However, the use of calculators for storing and/or recovering formulae or constants is NOT permitted.

Questions should be directed to the proctor.

Good luck!!

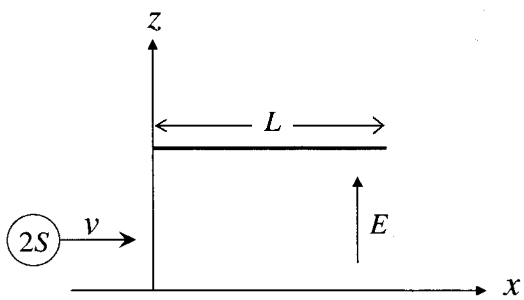
1. An electron beam is prepared by heating a filament, subjecting the emitted electrons to a potential difference V_0, and using a series of electrostatic lenses to guide the electrons on a trajectory parallel to the x-axis. A steel plate with two slits parallel to the y-axis is located at $x = 0$. The distance between the slits is d. The electrons are detected on a screen located at $x = X \gg d$. The screen is free to move along the z-axis. On the screen we observe an interference pattern. In principle, it is possible with this setup to determine through which slit the electron traveled. How? Does this destroy the interference pattern?

2. A beam of excited hydrogen atoms is prepared in the $2S$ state and passed between the plates of a capacitor in which a uniform electric field $\vec{E}$ exists over a distance L (see figure). The hydrogen atoms have velocity v and move parallel to the x axis in the $+x$ direction; the electric field is directed along the z axis. Assume that all $n = 2$ states of hydrogen are degenerate in the absence of the E-field (i.e. neglect hyperfine splittings). In the presence of the E-field, certain of the states will mix. You may neglect coupling to states of $n \neq 2$.

(a) Which of the $n = 2$ states are mixed by the E field to first order in E? Justify your statements by symmetry or other arguments.

(b) Write the Hamiltonian describing the time evolution of the $n = 2$ states for $0 < x < L$.

(c) For an atom which enters the capacitor at time $t = 0$ in the $2S$ state, find the wave function at time $t < L/v$.



(d) If the entering beam contains only atoms in the $2S$ state, find the probability that the emergent beam contains atoms in the $2S$, $2P_x$, $2P_y$ and $2P_z$ states.

Some possibly useful information:

$$\int_0^\infty dx x^n e^{-ax} = \frac{n!}{a^{n+1}}$$

Hydrogenic wave functions:

$$\psi_{1,0,0} = \frac{1}{\sqrt{\pi}}\left(\frac{1}{a_B}\right)^{3/2} e^{-\frac{r}{a_B}}$$

$$\psi_{2,0,0} = \frac{1}{4\sqrt{2\pi}}\left(\frac{1}{a_B}\right)^{3/2}\left(2 - \frac{r}{a_B}\right) e^{-\frac{r}{2a_B}}$$

$$\psi_{2,1,x} = \frac{1}{8\sqrt{2\pi}}\left(\frac{1}{a_B}\right)^{3/2} \frac{x}{a_B} e^{-\frac{r}{2a_B}}$$

$$\psi_{2,1,y} = \frac{1}{8\sqrt{2\pi}}\left(\frac{1}{a_B}\right)^{3/2} \frac{y}{a_B} e^{-\frac{r}{2a_B}}$$

$$\psi_{2,1,z} = \frac{1}{4\sqrt{2\pi}}\left(\frac{1}{a_B}\right)^{3/2} \frac{z}{a_B} e^{-\frac{r}{2a_B}}$$

3. The energy levels of an isotropic three-dimensional oscillator are easily found to be $E = (N+\frac{3}{2})\hbar\omega$ (where $N = N_x + N_y + N_z$ and N_x, N_y, N_z are the number of quanta in cartesian coordinates). All the degenerate states (fixed N) can also be classified in terms of orbital angular momentum $l = N,\ N-2,\ N-4,\ \ldots,\ 0$ for N even and $l = N,\ N-2,\ N-4,\ \ldots,\ 1$ for N odd.

Now suppose there is a small perturbation which breaks the rotational invariance of the system. In this problem you will determine the manner in which the perturbation breaks the degeneracy, for two forms of the perturbation:

$$H_{\text{pert.1}} = \alpha r^5 Y_{5,0}(\theta, \phi)$$

and

$$H_{\text{pert.2}} = \beta r^6 Y_{6,0}(\theta, \phi)$$

Hint: You do not need to evaluate any integrals to solve this problem.

(a) Show that the perturbation $H_{\text{pert.1}}$, when evaluated to first order in α, does not lead to a splitting of the levels of the degenerate multiplets (labeled by N).

(b) Show that the Hamiltonian including $H_{\text{pert.2}}$ (but not $H_{\text{pert.1}}$) has a spectrum for which m is a good quantum number. Moreover, show that for any level $m \neq 0$ there is a degenerate level with $-m$.

(c) Show that the perturbation due to $H_{\text{pert.2}}$, when evaluated to first order in β, does not lead to a splitting of any of the levels in the $n = 2$ multiplet.

(d) Show that for the $N = 3$ multiplet, $H_{\text{pert.2}}$, when evaluated to first order in β, has the following effects on the spectrum:

i. The $l = 1$ states are not split from each other (all m remain degenerate).

ii. Among the $l = 3$ states, the states with $m = \pm 3, \pm 2, \pm 1, 0$ are split from one another, and also from the $l = 1$ states.

4. Consider a spinless particle in a three-dimensional potential, with the Hamiltonian

$$H = \frac{\boldsymbol{P}^2}{2m} + \frac{k}{2} r^2$$

(a) Find the energy eigenvalues, and determine the degeneracy of the lowest four.

(b) Now suppose instead that five identical particles are in this potential. What is the ground state energy of this system if these particles have
(i) spin $\frac{1}{2}$, (ii) spin 1, or (iii) spin $\frac{3}{2}$? Assume that these particles do not interact with each other.

5. A static uniform electric field $\vec{E} = E\vec{x}$ is applied to an electron in a harmonic potential with natural frequency ω.

(a) Describe the effect of the field on the harmonic oscillator wave functions and on its spectrum.

(b) Calculate the induced dipole moment of the electron, and the polarizability (the ratio of the induced dipole moment to the field strength).

SOLUTIONS
Section 3
QM

Brooijmans - Quantum

Sec 3 QM
Problem # 1
Brooijmans

Quals 08. Quantum

December 18, 2008

Problem

An electron beam is prepared by heating a filament , submitting the emitted electrons to a potential difference V_0, and using a series of electrostatic lenses to guide the electrons on a trajectory parallel to the x-axis. A steel plate with two slits parallel to the y-axis is located at $x = 0$. The distance between the slits is d. The electrons are detected on a screen located at $x = X \gg d$, which is free to move along the z-axis. On the screen we observe an interference pattern. In principle, it is possible with this setup to determine through which slit the electron traveled. How? Does this destroy the interference pattern?

Solution

To be able to distinguish the interference pattern on the screen we need to be able to measure its position with precision better than the distance between the peaks, so

$$\sigma_{pos} < a\frac{\lambda X}{2d} = a\frac{hX}{2p_0 d} \tag{1}$$

where p_0 is the electron's momentum and a is a positive number smaller than 1. Let us take $a = 1/3$.

When an electron hits the screen at a point $z \neq \pm d/2$, its momentum vector is not parallel with the x-axis so it will transfer momentum to the screen which is free to move along the z-axis. Let θ_1 be the angle between the top slit and the point at z:

$$\theta_1 \simeq (z - d/2)/X, \tag{2}$$

then the momentum transfered to the screen if the electron passed through slit 1 is $q_1 \simeq -p_0\theta_1$ (since θ_1 is small). Similarly, $q_2 \simeq -p_0\theta_2$ with $\theta_2 \simeq$

$(z + d/2)/X$. So if we can measure the momentum of the screen along the z-axis well enough to distinguish q_1 from q_2 we can determine through which slit the electron passed.

To do that, our resolution on the momentum needs to be

$$\sigma_{mom} < b(q_1 - q_2) = bp_0 \frac{d}{X} \tag{3}$$

with b a number less than one. Let's again take $1/3$. To see the interference pattern **and** determine which slit the electron went through we need

$$\sigma_{pos}\sigma_{mom} < ab\frac{hX}{2p_0 d} p_0 \frac{d}{X} = ab\frac{h}{2}, \tag{4}$$

and we see that this violates the uncertainty principle.

Millis
Sec 3 QM
Problem # 2

Millis Quals 08 Quantum Solution

(a): Label the $n = 2$ states as $2S$, $2P_x$, $2P_y$, $2P_z$. The perturbing potential is $V(x, y, z) = Ez$, so over the scale of the atom is a constant term (shifting all levels) and a term odd under reflection in z. Rotation invariance about z ensures that $2P_x$ and $2P_y$ remain decoupled. The perturbation therefore mixes $2S$ and $2P_z$.

(b) The atom moves on the line $z = z_i$. On this line the potential is

$$V = Ez_0 + E(z - z_0)$$

In the basis $(2S, 2P_z, 2P_x, 2P_y)$ the Hamiltonian is

$$H = \begin{pmatrix} E & M & 0 & 0 \\ M & E & 0 & 0 \\ 0 & 0 & E & 0 \\ 0 & 0 & 0 & E \end{pmatrix}$$

with

$$E = \frac{13.6eV}{4} + Ez_0$$

and

$$M = E\int_0^{2\pi} d\phi \int_{-1}^{1} dcos\theta \int_0^{\infty} r^2 dr \frac{1}{32\pi a_B^3} \frac{z^2}{a_B}\left(2 - \frac{r}{a_B}\right) e^{-\frac{r}{a_B}}$$

adopting dimensionless coordinates we find

$$M = \frac{Ea_B}{24}\int_0^{\infty} u^2 du u^2 (2 - u) e^{-u} = \frac{Ea_B}{24}(2 * 4! - 5!) = -3Ea_B \qquad (1)$$

(c) The eigenfunctions are $\psi_{\pm} = \frac{1}{\sqrt{2}}(|2S> \pm|2P_z>)$ with energy $E \pm M$. The initial state is $|2S> = \frac{1}{\sqrt{2}}(|\psi_+> + |\psi_->)$.Thus at time t we have

$$\psi(t) = e^{iEt}\frac{1}{\sqrt{2}}\left(e^{iMt}|\psi_+> + e^{-iMt}|\psi_->\right) = e^{iEt}\left(cos(Mt)|2S> + sin(Mt)|2P_z>\right)$$

(d) The beam has probability $cos(ML/v)^2$ to be in the $2S$ state and $sin(ML/v)^2$ to be in the $2P_z$ state and no probability to be in any of the other states.

Sec. 3 QM
Problem # 2
Millis

Millis Quals 08 Possibly Useful Information

$$\int_0^\infty dx x^n e^{-ax} = \frac{n!}{a^{n+1}}$$

Hydrogenic wave functions

$$\psi_{1.0.0} = \frac{1}{\sqrt{\pi}}\left(\frac{1}{a_B}\right)^{3/2} e^{-\frac{r}{a_B}}$$

$$\psi_{2.0.0} = \frac{1}{4\sqrt{2\pi}}\left(\frac{1}{a_B}\right)^{3/2}\left(2-\frac{r}{a_B}\right) e^{-\frac{r}{2a_B}}$$

$$\psi_{2.1.x} = \frac{1}{8\sqrt{2\pi}}\left(\frac{1}{a_B}\right)^{3/2}\frac{x}{a_B} e^{-\frac{r}{2a_B}}$$

$$\psi_{2.1.y} = \frac{1}{8\sqrt{2\pi}}\left(\frac{1}{a_B}\right)^{3/2}\frac{y}{a_B} e^{-\frac{r}{2a_B}}$$

$$\psi_{2.1.z} = \frac{1}{4\sqrt{2\pi}}\left(\frac{1}{a_B}\right)^{3/2}\frac{z}{a_B} e^{-\frac{r}{2a_B}}$$

Poulin QM
SEC 3
Problem # 3

Soln Angular mom.

In general, diagonalize perturbation in each degenerate subspace, but here simpler.

For given $N = N_x + N_y + N_z$:

$\ell = N, N-2, \ldots, 0$ if N even

$\ell = N, N-2, \ldots, 1$ if N odd

a) $\langle N, \ell', m' | Y_{5,0} | N, \ell, m \rangle = 0$ since $\ell - \ell'$ even (can't talk to 5)

or also $|N, \ell', m'\rangle$ & $|N, \ell, m\rangle$ have same parity $(-1)^{\ell'} = (-1)^{\ell}$

but $Y_{5,0}$ is parity odd.

b) Since $\langle N, \ell', m' | Y_{6,0} | N, \ell, m \rangle \propto \delta_{mm'}$ $\longrightarrow$ perturbation commutes with L_z

Can also use time-reversal $Y_{6,0} \xrightarrow{T} (-1)^6 Y_{6,0} = Y_{6,0}$

$\Rightarrow [H_0 + H_{pert.2}, T] = 0$

But $|\ell, m\rangle \xrightarrow{T} |\ell, -m\rangle$ $\Rightarrow$ $|m\rangle$ & $|-m\rangle$ degenerate

c) $N = 2$ has $\ell = 0, 2$.

Wigner - Eckart Th. : $\langle \ell', m | Y_{k,0} | \ell, m \rangle \neq 0$ only if $|\ell - k| \leq \ell' \leq \ell + k$

$\Rightarrow$ vanishes for $k = 6$, $\ell, \ell' = 0, 2$.

d) For $N = 3$ $\Rightarrow$ $\ell = 1, 3$, but $Y_{6,0}$ does not connect $\ell = 1$ with $\ell = 3$

$\Rightarrow$ can treat $\ell = 1$ & $\ell = 3$ seperately

i) $\langle 1, m' | Y_{6,0} | 1, m \rangle = 0$ $\Rightarrow$ no energy shift from $H_{pert.2}$

ii) Wigner - Eckart : radial integral $\neq 0$ (m-independent)

$$\langle 3, m' | Y_{6,0} | 3, m \rangle = A \, \delta_{mm'} \underbrace{\langle 3, m | k=6, q=0, \ell=3, m \rangle}_{\text{Clebsch-Gordan coeff: } \neq 0, \text{ m-dependent}}$$

$\Rightarrow$ m-dependent energy shift for $\ell = 3$.

Weinberg 2

a) $H = \frac{\vec{p}^2}{2m} + \frac{1}{2}kr^2$

$$= \left(\frac{p_x^2}{2m} + \frac{1}{2}kx^2\right) + \left(\frac{p_y^2}{2m} + \frac{1}{2}ky^2\right) + \left(\frac{p_z^2}{2m} + \frac{1}{2}kz^2\right)$$

Energy levels $= \hbar\omega(n_x + n_y + n_z + 3/2)$, $\omega = \sqrt{\frac{k}{m}}$

$n = 0, 1, 2 \ldots$

Lowest levels $(N = n_x + n_y + n_z)$

$N = 0$	$E = 3/2\,\hbar\omega$	deg $= 1$
$N = 1$	$5/2\,\hbar\omega$	3
$N = 2$	$7/2\,\hbar\omega$	$6 = \frac{(3\times 4)}{2}$
$N = 3$	$9/2\,\hbar\omega$	$10 = \frac{(3\times 4\times 5)}{3!}$

b) (i) Spin 1/2 → two spin states for each $\{n_x, n_y, n_z\}$
⇒ ground state has $N = 0, 0, 1, 1, 1$
⇒ $E = \frac{21}{2}\hbar\omega$

(ii) Spin 1 → bosons, all in lowest state
⇒ $E = 15/2\,\hbar\omega$

(iii) Spin 3/2 → four spin states for each $\{n_x, n_y, n_z\}$
⇒ ground state has $N = 0, 0, 0, 0, 1$
⇒ $E = 17/2\,\hbar\omega$

Sec 3 QM
Problem # 5
Zelevinsky

MODERN PHYSICS – QUANTUM MECHANICS

Polarizability. SOLUTION.

The Hamiltonian of the harmonic oscillator including the field is

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2 - e\mathcal{E}x = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 (x - x_0)^2 - \frac{e^2\mathcal{E}^2}{2m\omega^2}. \tag{1}$$

where

$$x_0 = \frac{e\mathcal{E}}{m\omega^2}. \tag{2}$$

a) From Eq. (1), it follows that the field shifts the equilibrium position of the oscillator to x_0, and shifts the spectrum as a whole by $(-e^2\mathcal{E}^2/(2m\omega^2))$.

b) The induced dipole moment along $\hat{x}$ is given by the displacement x_0:

$$\langle d \rangle = \langle -ex \rangle = -ex_0 = -\frac{e^2\mathcal{E}}{m\omega^2}, \tag{3}$$

which corresponds to the polarizability

$$\alpha \equiv \frac{d}{\mathcal{E}} = -\frac{e^2}{m\omega^2}. \tag{4}$$