Columbia University
Department of Physics
QUALIFYING EXAMINATION
Friday, January 18, 2008
11:10 AM – 1:10 PM

General Physics (Part II)
Section 6.

Two hours are permitted for the completion of this section of the examination. Choose 4 problems out of the 6 included in this section. (You will not earn extra credit by doing additional problems). Apportion your time carefully.

Use separate answer booklet(s) for each question. Clearly mark on the answer booklet(s) which question you are answering (e.g., Section 6 (General Physics), Question 3; Section 6(General Physics) Question 6, etc.).

Do **NOT** write your name on your answer booklets. Instead clearly indicate your **Exam Letter Code**.

You may refer to the single handwritten note sheet on 8 ½ x 11" paper (double-sided) you have prepared on General Physics. The note sheet cannot leave the exam room once the exam has begun. This note sheet must be handed in at the end of today's exam. Please include your Exam Letter Code on your note sheet. No other extraneous papers or books are permitted.

Simple calculators are permitted. However, the use of calculators for storing and/or recovering formulae or constants is NOT permitted. Questions should be directed to the proctor.

Good luck!!

1. A black hole of mass M can be thought of, schematically, as a spherical region of radius $r_M = 2GM$ (working in $c = 1$ units) where nothing that enters can escape. r_M is known as the Schwarzschild radius. Also, whenever some mass (or energy) M is confined within a sphere of radius r_M, the mass undergoes gravitational collapse and becomes a black hole with the same mass.

Consider a gas of black holes, all with identical masses M, uniformly distributed inside a large cubic box with volume $V = L^3$. Assume that the black holes have negligible velocities.

(a) Show that the black-hole gas can never be so dense as to have the typical distance between two nearby black holes of the order of their Schwarzschild radius.

(b) For a given L and M, estimate the maximum allowed number density of black holes.

[For both questions ignore all numerical factors like 2 and π.]

2. Consider a system of non-interacting particles of spin ½. Each particle has mass m, and is constrained to move with non-relativistic momentum on the very flat surface of superfluid liquid He kept at temperature $T \to 0$. The components of the particle momentum are restricted to the (x,y) plane that defines the surface of the liquid. The interactions between particles and the liquid He substrate are negligible. The area is $A = L^2$, where L is the length of the surface along the x- and y-directions. The number of particles per unit area is n.

(a) Obtain the energy levels E, assuming cyclic boundary conditions. [In cyclic boundary conditions the wavefunctions at position (x,y) are identical to those at positions $(x+L, y)$, $(x, y+L)$, and $(x+L, y+L)$].

(b) Use the results in (a) to obtain the expression for the density of states $g(E)$. Note that $g(E)dE$ gives the number of single-particle energy levels in the range between E and $E+dE$.

(c) Consider the limit of temperature $T = 0$. Obtain the energy difference between the lowest and the highest energy states that are occupied by the particles.

(d) What is the average energy per particle at $T = 0$?

(e) The temperature is raised slightly, so that ΔT is much smaller than the average energy per particle. Describe in words the changes that occur in the system. What is the expected temperature dependence of the change in the total energy of the system?

3. Estimate very roughly, from the heat of vaporization of water (L ~ 2×10^3 Joules/gram), Avogadro's number (N_A ~ 6×10^{23}) and whatever natural constants you might need:

(a) The surface tension of water;

(b) The speed of very small wavelength (λ) ripples on an otherwise flat surface of water;

(c) The lowest frequency (density preserving) oscillation of a drop of water with mass M;

(d) The radius r of capillary tubes inside a tree which bring water from the tree's roots to its leaves 30m higher up.

4. Consider a spherical (radius R) container enclosing very hot (temperature T) fully ionized hydrogen with total mass M.

(a) Estimate roughly the total photon energy contained by the sphere.

(b) About how long will it take for most of the photons of part (a) to escape due to diffusion of photons to the sphere surface? Assume that photons interact with the ionizing hydrogen only by Thomson scattering on free electrons.

(c) Use (a) and (b) to estimate the photon luminosity (L) from the sphere.

(d) Suppose the sphere is held together by gravitational attraction of the hydrogen and kept from collapsing by the kinetic motion of the electrons and protons. Show that

$$L = AM^{\alpha}$$

with A and the exponent α independent of R, T and M. Give A and α in terms of the "other constants".

[Express results in terms of M, R, T and other constants from among electron mass (m_e), hydrogen mass (m_H), Boltzmann constant (k_B), electron charge (e), Planck constant/2π ($\hbar$), speed of light (c), and gravitational constant (G).]

5. Consider a 3-dimensional metal, like Na, where one conduction electron comes from one atom. The crystal structure has one atom per unit cell. The lattice constant of this system is about 3 Angstroms, and the melting temperature is about 700 K. The atomic weight is about 50. (These numbers are given just to qualitatively understand the situation. It may not be necessary to use them in answering the following questions.)

Consider the specific heat of this system.

(a) Show that the specific heat from phonons is proportional to T^3 at low temperatures (Debye model). Estimate the minimum wavelength and hence maximum wavenumber of phonons in the metal.

(b) Show that the specific heat from the conduction electrons is proportional to T at low temperatures. We assume that the effective mass of electrons in this metal is equal to that of a bare electron.

(c) At T = 300 K, which one is larger: the specific heat from the lattice or that from the conduction electrons? Describe your reasoning.

(d) When the effective mass of the electrons is 100 times that of the bare electron, how does the specific heat from the electron system change from the value for the bare electron mass?

(e) In the so-called Einstein model, where all phonons are taken to have the same characteristic frequency ω_0, derive the temperature dependence of the specific heat in the limit of low temperature.

6. The flux of solar radiation incident on the earth's atmosphere is 1370 Watts per square meter. About 30% of this is reflected immediately; the remainder is absorbed by the Earth, leading to an effective solar constant of roughly 960 W/m^2. The radiation is peaked in the visual range of frequencies.

(a) Calculate the mean temperature of the earth neglecting the effects of the atmosphere.

(b) Model the atmosphere as a single thin layer which is transparent to the solar radiation, but which totally reabsorbs the (infrared) frequencies emitted by the Earth. Calculate the mean surface temperature in the presence of such a layer.

(c) Assume the density of infrared absorbing gases in the atmosphere doubles. Model this as two thin layers, and calculate the mean surface temperature.

The Stefan-Boltzmann constant $\sigma = 5.67 \times 10^{-8}\ \mathrm{W\ m^{-2}\ K^{-4}}$.

Sec 6, #1
Nicolis

2. A black hole of mass M can be thought of, schematically, as a spherical region of radius $r_M = 2GM$ (we work in $c = 1$ units) from where nothing that enters can escape. r_M is known as the Schwarzschild radius. Also, whenever some mass (or energy) M is confined within a sphere of radius r_M, the mass undergoes gravitational collapse and becomes a black hole with the same mass.

Consider a gas of black holes, all with identical masses M, uniformly distributed inside a large cubic box with volume $V = L^3$. Assume that the black holes have negligible velocities.

8 *(a)* Show that the black-hole gas can never be so dense as to have the typical distance between two nearby black holes of order of their Schwarzschild radius.

7 *(b)* For given L and M, estimate the maximum allowed number density of black holes.

[For both questions ignore all numerical factors like 2 and π]

Solution.

(a) The problem in making the gas very dense is that when we bring many black holes together they tend to collapse into a larger black hole. Indeed if we have $N \gg 1$ of them, their overall mass is NM, with a corresponding Schwarzschild radius $r_{NM} = Nr_M$. On the other hand, if we imagine that their typical distance is of order of their radius r_M, they all fit in a sphere of radius

$$R \sim N^{1/3} r_M \ll N r_M = r_{NM} \,. \tag{5}$$

Hence, well before they can get so close to one another, they all collapse into a larger black hole.

(b) Call n the black-hole number density. In a sphere of radius R there are $\sim nR^3$ black holes, their total mass is $\sim nR^3 M$, the corresponding Schwarzschild radius is $\sim nR^3 r_M$. For the whole sphere not to collapse into a black hole we want that its radius be larger than its Schwarzschild radius,

$$R \gtrsim nR^3 r_M \,, \tag{6}$$

from which we immediately get $n \lesssim 1/(R^2 r_M)$. The r.h.s. depends on the radius of the sphere we consider. The bound is more stringent for larger R. The absolute bound is thus obtained by taking the largest possible sphere, $R \sim L$,

$$n_{\max} \sim \frac{1}{L^2 r_M} \,. \tag{7}$$

That is, the typical distance among nearby black holes is always parametrically larger than r_M, and at best of order $d_{\min} \sim (L^2 r_M)^{1/3}$.

Section 6 - General II Problem #2

(a) Quantization of momentum yields

$$p_x = \hbar \frac{2\pi n_x}{L} \; ; \quad p_y = \hbar \frac{2\pi n_y}{L}$$

where n_x and n_y are positive or negative integers.

The energy levels are:

$$E(n_x, n_y) = \frac{\hbar^2}{2m}\left(\frac{2\pi}{L}\right)^2 \left[n_x^2 + n_y^2\right]$$

(b)

$$g(E) = \frac{m}{\pi \hbar^2} \quad \text{(including spin degeneracy)}$$

independent of energy

(c) The number of states per unit area is:

$$dn = g(E)\, dE$$

$$n = \frac{m}{\pi \hbar^2} E_F$$

$$E_F = \frac{\pi \hbar^2 n}{m}$$

(d) $$\langle E \rangle = \int_0^{E_F} E\, g(E)\, dE$$

$$\langle E \rangle = \frac{m}{2\pi\hbar^2} E_F^2$$

From (c) $$\frac{m}{\pi\hbar^2} = \frac{n}{E_F}$$

$$\langle E \rangle = \frac{1}{2}\frac{n}{E_F} E_F^2 = \frac{1}{2} n E_F$$

$$\langle \varepsilon \rangle = \frac{1}{2} E_F$$

(e) The number of particles that change energy is proportional to ΔT

The typical energy change per particle is also proportional to ΔT

The total change in energy is proportional to $(\Delta T)^2$

$$\Delta \varepsilon \sim (\Delta T)^2$$

Section 6, # 3

General Physics — Mal Ruderman

Problem 1

Estimate very roughly, from the heat of vaporization of water ($L \sim 2 \times 10^3$ joules/gram), Avagadro's number ($N_A \sim 6 \times 10^{23}$), and whatever ~~other~~ natural constants you might need

a) the surface tension of water ;

b) the speed of very small wavelength (λ) ripples on an otherwise flat surface of water ;

c) the ~~first~~ lowest frequency (density preserving) oscillation of a drop of water with mass M ;

d) the radius (r) of capillary tubes inside a tree which bring water from the tree's roots to its leaves 30 meters higher up .

Section 6, #4

General
Problem #2
2.

Mal Ruderman

How bright are typical stars?
Consider a spherical (radius R) container enclosing ~~the~~ very hot (temperature T) fully ionized hydrogen with total mass M.

~~a)~~ [Express results below in terms of M, R, T and other constants from among ~~the~~ electron mass (m_e) hydrogen mass (m_H), Boltzmann constant (k_B), electron charge (e), Planck constant/2π ($\hbar$), speed of light (c), and gravitational constant (G).]

a) Estimate roughly the total ~~contained~~ photon energy contained by the sphere.

b) About ~~How~~ how long will it take for most of the photons of a) to escape if this happens to any photon which diffuses to the sphere surface? Assume that diffusing photons interact with the hydrogen they diffuse through only by Thomson scattering on free electrons

c) Use a) and b) to estimate the photon luminosity (L) from the sphere

General Physics
Problem #2 page 2

a) If the sphere is held together by gravitational attraction of the hydrogen and kept from collapsing by the kinetic motion of the electrons and protons

$$L = A M^{\alpha}$$

with A and the exponent α independent of R, T, and M. Give ~~these~~ in terms of the "other constants".

A and α

General Problem #2 Mal Ruderman
Answers

a) $E_{photons} \sim R^3 k_B T \left(\frac{k_B T}{\hbar c}\right)^3$

b) $time\,(t) \sim \frac{R^2}{\lambda_{mfp} c}$

$\lambda = \frac{1}{n_e \sigma_T}$ $n_e = \frac{M}{m_H}$

$\boxed{\sigma_T \sim (e^2/mc^2)^2}$

$\therefore t \sim \frac{M \sigma_T}{c m_H R} \sim \left(\frac{e^2}{mc^2}\right)^2 \left(\frac{M}{m_H}\right)\left(\frac{1}{cR}\right)$

c) $L \sim \frac{E_{photons}}{t}$

d) $\frac{GM^2}{R} \sim k_B T \times \frac{M}{m_H}$

$L \sim \frac{G^4 m_H^5}{\hbar^3 c^2 \sigma_T} M^3$

$\therefore \alpha = 3, \quad A \approx \frac{G^4 m_H^5 m_e^2 c^2}{\hbar^3 e^4}$

III. Part 6
Prob. #5.

Tomo Uemura

(a) Phonon $\Rightarrow$ Bose Stat.

Linear Dispersion $\varepsilon = ck$.

$D(k) \sim k^2 dk \quad D(\varepsilon) \sim \varepsilon^2 d\varepsilon$

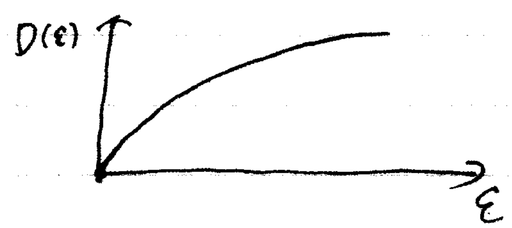
$$n \propto \frac{1}{\exp\left(\frac{\varepsilon}{kT}\right) - 1}$$

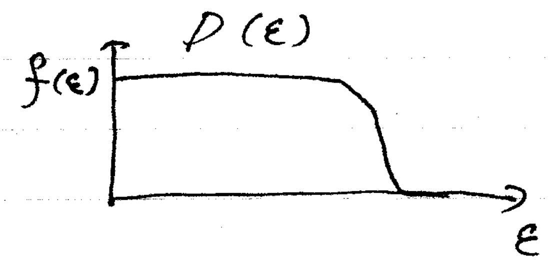
for low-T $\quad n \sim \exp\left(-\frac{\varepsilon}{kT}\right)$

$$\langle E \rangle \sim \int \varepsilon D(\varepsilon)\, d\varepsilon$$
$$\sim \int \varepsilon^3 \exp\left(-\frac{\varepsilon}{kT}\right) d\varepsilon$$
$$\propto T^4$$

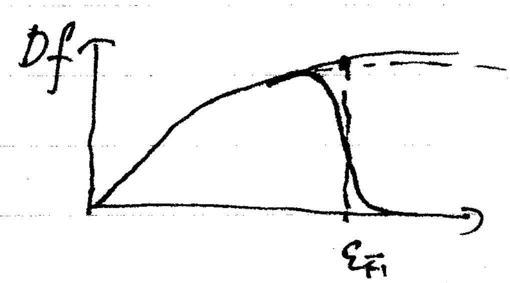
$$C = \frac{d\langle E \rangle}{dT} \sim T^3$$

(b) Electrons : Fermions.





$f(\varepsilon)$: Fermi dist.



Excitable states in width $\sim kT$
each state carries energy $\sim kT$

$\langle \varepsilon \rangle \propto T^2 \qquad C \propto T$

(c). In the high-T limit, both C_{ph} and C_{el} have classical value for a gas.

Crossover to classical region characterized by $\Theta_{Debye} \sim \Theta_{melting} \sim 700°K$ for phonons

Fermi Temp for electrons ↳ order 10^4 K.

At room temp, electron system is still degenerated.

Thus $C_{phonon} \gg C_{electron}$.

(d). $C_{electron}$ or γ Sommerfeld Coeff

$\gamma \propto D(\varepsilon_F) \propto m^*$

∴ 100 times larger, scaling with m^*

(e) $\langle E \rangle \sim n\hbar\omega_0$

at low-T $\langle E \rangle \sim \dfrac{\hbar\omega}{1 - e^{-\hbar\omega/kT}}$ $\sim \hbar\omega \cdot \exp\left(\frac{\hbar\omega}{kT}\right)$ @ low T

$$C \sim \boxed{\frac{d\langle E \rangle}{dT} \sim \frac{\exp\left(-\frac{\hbar\omega}{kT}\right)}{T^2}}$$

Zajc

Problem: The flux of solar radiation incident on the earth's atmosphere is 1370 Watts per square meter. About 30% of this is reflected immediately; the remainder is absorbed by the earth, leading to an effective solar constant of roughly 960 W/m^2. The radiation is peaked in the visual range of frequencies. a) Calculate the mean temperature of the earth neglecting the effects of the atmosphere. b) Model the atmosphere as a single thin layer which is transparent to the solar radiation, but which totally reabsorbs the (infrared) frequencies emitted by the earth. Calculate the mean surface temperature in the presence of such a layer. c) Assume the density of infrared absorbing gases in the atmosphere doubles. Model this as two thin layers, and calculate the mean surface temperature.

The Stefan-Boltzmann constant $\sigma = 5.67 \times 10^{-8}\text{W m}^{-2}\text{ K}^{-4}$.

Solution: a) The earth intercepts radiation with cross section πR^2 and re-radiates with cross section $4\pi R^2$, so in the absence of atmospheric heat blankets

$$\frac{1}{4}\Phi_{eff} = \sigma T^4 \Rightarrow 240\text{ W/m}^2 = 5.67 \times 10^{-8}\text{W m}^{-2}\text{ K}^{-4} T^4 \Rightarrow T \approx 255\text{ K} \quad .$$

b) You can solve this by drawing a picture and counting photons, which is mathematically equivalent to

$$\Phi_0 = \Phi_1 \tag{1}$$
$$\Phi_0 + \Phi_1 = \Phi_S \quad , \tag{2}$$

where $\Phi_0 \equiv \Phi_{eff}/4$ (as in part "a"), Φ_S is the flux leaving the earth's surface and Φ_1 is the flux from one side of the thin layer. This immediately gives $\Phi_S = 2\Phi_0$, or

$$T = 2^{1/4} \times 255\text{ K} \approx 303\text{ K}$$

(a fairly reasonable estimate).

c) Repeating, with the outermost layer labeled as "2" :

$$\Phi_0 = \Phi_2 \tag{3}$$
$$\Phi_S + \Phi_2 = 2\Phi_1 \tag{4}$$
$$\Phi_0 + \Phi_1 = \Phi_S \quad , \tag{5}$$

giving $\Phi_S = 3\Phi_0$, or

$$T = 3^{1/4} \times 255\text{ K} \approx 336\text{ K}$$

(!).