Columbia University
Department of Physics
QUALIFYING EXAMINATION
Wednesday, January 16, 2008
9:00 AM – 11:00 AM

Modern Physics
Section 3. Quantum Mechanics

Two hours are permitted for the completion of this section of the examination. Choose 4 problems out of the 5 included in this section. (You will not earn extra credit by doing an additional problem). Apportion your time carefully.

Use separate answer booklet(s) for each question. Clearly mark on the answer booklet(s) which question you are answering (e.g., Section 3 (QM), Question 1; Section 3(QM) Question 5, etc.).

Do **NOT** write your name on your answer booklets. Instead clearly indicate your **Exam Letter Code**.

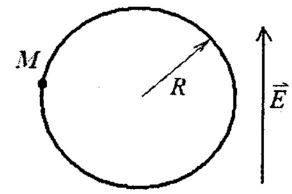
You may refer to the single handwritten note sheet on 8 ½ x 11" paper (double-sided) you have prepared on Modern Physics. The note sheet cannot leave the exam room once the exam has begun. This note sheet must be handed in at the end of today's exam. Please include your Exam Letter Code on your note sheet. No other extraneous papers or books are permitted.

Simple calculators are permitted. However, the use of calculators for storing and/or recovering formulae or constants is NOT permitted.

Questions should be directed to the proctor.

Good luck!!

1. A particle of mass M and charge q is constrained to move in a circle of radius R.



(a) If no forces other than those of constraint act on the particle, find its allowed energies and the corresponding eigenstates.

(b) A strong, uniform electric field $\vec{E}$, oriented in the plane of the circle, is applied to the system. Find the first few lowest eigenvalues and corresponding eigenstates. Assume that $qRE >> \hbar^2/MR^2$.

(c) If a uniform magnetic field $\vec{B}$ is applied perpendicular to the plane of the circle, find the resulting eigenvalues and eigenstates. Work this out for both the $\vec{E} = 0$ situation of part (a) as well as the $\vec{E} \neq 0$ case of part (b).

2. Consider two identical, non-interacting spin ½ particles of mass m. The particles are moving in a potential given by

$V(x_1, x_2) = 0$	for x_1 and x_2 in the interval $(-a/2, a/2)$
$V(x_1, x_2) = \infty$	otherwise

This is just the conventional 1-dimensional particle(s) in a box.

(a) For the singlet spin state, explicitly solve the Schrödinger equation to obtain the 2 particle ground state wavefunction. Calculate the energy of the ground state.

(b) For the triplet spin state, explicitly solve the Schrödinger equation to obtain the 2-particle ground state wavefunction. Calculate the energy of the ground state.

Assume that an interaction potential between the particles is turned on. The form of the interaction is $v = v_o b\, \delta(x_1 - x_2)$ where v_o is the interaction strength and b a characteristic length.

(c) Calculate the first order correction to the ground state energy in the triplet and singlet states.

(d) Provide a simple physical explanation for the magnitude of the first order correction for the singlet and triplet states.

3. Consider a single-electron atom within the field of a laser producing an oscillating local electric field of $\mathcal{E}(t) = \mathrm{Re}[\mathcal{E}_0 \exp(-i\omega t)]$ in the z-direction. Treating the laser semi-classically, we can write the perturbation acting on the atom's electron (within the dipole approximation) as $V = -e\,\mathcal{E}(t)\,z$, where $-e$ is the charge of the electron.

We now tune the laser to be exactly resonant with a transition between the ground state $|0\rangle$ and excited state $|1\rangle$ of the atom, i.e., the energy difference between the two states matches the photon energy of the laser: $E_1 - E_0 = \hbar\omega$. In the following, treat the atom as a (non-degenerate) two-level system. *Do not use perturbation theory, but do neglect any weak, rapidly varying terms in the response.*

(a) Find the probability that the atom is in state $|1\rangle$ as a function of time t. Take the atom to be in state $|0\rangle$ at $t = 0$.

(b) For a typical (allowed) optical transition in the atom, estimate the electric field strength needed to achieve the maximum population of state $|1\rangle$ in 10^{-9} s. If this electric field is to be provided by a laser beam with a power of 1 mW, how tightly should the laser be focused?

Hint: Write the solution of the time-dependent Schrödinger equation as

$$|\psi\rangle = C_0(t)\exp(-iE_0t/\hbar)|0\rangle + C_1(t)\exp(-iE_1t/\hbar)|1\rangle$$

and find equations for the time-dependent coefficients.

4. Consider a simple harmonic oscillator with frequency ω. A coherent state $|\lambda\rangle$ is defined to be an eigenstate of the lowering operator: $\hat{a}|\lambda\rangle = \lambda|\lambda\rangle$. Since the lowering operator isn't Hermitian, λ can be a complex number.

(a) Show that, up to an overall normalization, a coherent state can be expressed as

$$|\lambda\rangle = \exp(\lambda\hat{a}^\dagger)|0\rangle$$

Here $\hat{a}^\dagger$ is the raising operator and $|0\rangle$ is the ground state.

(b) Compute the position-space wavefunction $\psi(x) = \langle x|\lambda\rangle$ for a coherent state. Hint: derive a differential equation for $\psi(x)$.

(c) Compute the normalized expectation values of the position and momentum operators

$$\frac{\langle\lambda|\hat{x}|\lambda\rangle}{\langle\lambda|\lambda\rangle} \qquad \frac{\langle\lambda|\hat{p}|\lambda\rangle}{\langle\lambda|\lambda\rangle}$$

(d) Start with a coherent state $|\lambda_0\rangle$ at time t = 0. Show that up to an overall phase, under time evolution this state evolves into a coherent state $|\lambda(t)\rangle$. Express $\lambda(t)$ in terms of λ_0.

Useful facts: $\hat{a} = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} + \frac{i\hat{p}}{m\omega}\right)$, $\hat{a}^\dagger = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} - \frac{i\hat{p}}{m\omega}\right)$

5. Consider two identical particles of mass m and spin $S = ½$ with a spin-dependent interaction between them. Let $r = r_1 - r_2$ be the distance between the two particles and neglect any center of mass motion. The potential is

$$V(r) = \frac{g^2}{r}\vec{\sigma}_1 \cdot \vec{\sigma}_2$$

with $\vec{\sigma}$ the usual trio of Pauli matrices.

(a) Prove that the total spin $\vec{S} = \vec{\sigma}_1 + \vec{\sigma}_2$ commutes with the Hamiltonian and find the eigenvalues of $\vec{\sigma}_1 \cdot \vec{\sigma}_2$

(b) Find the energies of all of the bound states.

(c) Find the possible values of the angular momentum for each of the bound states.

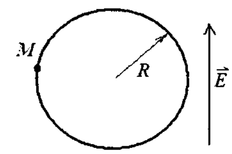
Section 3, QM, # 1
"QM round circle"

N. Christ

November 27, 2007

Quals Problems

1. A particle of mass M and charge q is constrained to move in a circle of radius R.



   (a) If no other forces other than those of constraint act on the particle, find its allowed energies and the corresponding eigenstates.

   (b) A strong, uniform electric field $\vec{E}$, oriented in the plane of the circle, is applied to the system. Find the first few lowest eigenvalues and corresponding eigenstates. Assume that $qRE \gg \hbar^2/MR^2$.

   (c) If a uniform magnetic field $\vec{B}$ is applied perpendicular to the plane of the circle, find the resulting eigenvalues and eigenstates. Work this out for both the $\vec{E} = 0$ situation of part (a) as well as the $\vec{E} \neq 0$ case of part (b). [You should neglect the $\vec{A}^2$ term in the Hamiltonian.]

2. A particle of mass M moves in a three-dimensional harmnic oscillator well. The Hamiltonian for this system is

$$H = \frac{\vec{p}^2}{2M} + \frac{1}{2}k\vec{r}^2$$

   (a) Find the energy and orbital angular momentum of the ground state and the first three excited states.

   (b) If eight identical, non-interacting spin-1/2 particles are placed in such a harmonic potential, find the ground state energy for the eight particle system.

   (c) Assume that these particles each have a magnetic moment $\vec{\mu} = \gamma\vec{s}$ where $\vec{s}$ is the particle's spin. If a magnetic field B is applied, what is the approximate ground state energy of the eight-particle system as a function of B? Plot the magnetization $(-\partial E/\partial B)$ for the ground state as a function of B.

Christ Section 3 QM
#1

Suggested Solutions

1. (a) $\psi(\theta) = \frac{1}{\sqrt{2\pi}} e^{im\theta}$, $m \in Z$ and $E_m = \frac{\hbar^2 m^2}{2MR^2}$.

   (b) For such a large E, the particle will under go simple harmonic motion around the minimum of the potential: $\psi(\theta)_n = h_n(z)e^{-z^2}$ where $z = \sqrt{\frac{M\omega}{\hbar}} R\theta$ and $E_n = \hbar\omega(n + \frac{1}{2})$. Here $\omega = \sqrt{\frac{qE}{MR}}$.

   (c) If $E = 0$, then the eigenstates are not changed but the energies are shifted by the usual magnetic moment–B coupling: $E_m = \frac{\hbar^2 m^2}{2MR^2} - \frac{e\hbar m B}{2Mc}$. For the $\vec{E} \neq 0$ case the effects of the magnetic field can be removed by adding a phase $e^{-i\frac{eBR^2}{2c\hbar}\theta}$ to the wave function and the energies are not affected.

2. (a) The ground state $\psi_0(\vec{r}) = Ne^{-\frac{m\omega}{\hbar}\vec{r}^2}$ and has energy $E_0 = \hbar\omega\frac{3}{2}$ with $\omega = \sqrt{k/m}$. This state has $l = 0$ and $m_l = 0$. The first three excited states all have energy $E_1 = \hbar\omega(1 + \frac{3}{2})$, $l = 1$ and $m_l = \pm 1$ and 0 if written:

$$(x \pm iy)e^{-\frac{m\omega}{\hbar}\vec{r}^2} \quad \text{and} \quad ze^{-\frac{m\omega}{\hbar}\vec{r}^2}.$$

   (b) The exclusion principle permits us to put two spin-1/2 particles in each of these four lowest states. The resulting energy is then $E_{\text{gnd}} = 2E_0 + 6E_1 = 18\hbar\omega$.

   (c) Since the particle spins are paired, for small B, there is no dependence of E_{gnd} on B. However, when $2\mu B$ becomes greater than the energy separation $\hbar\omega$ between the harmonic oscillator states, it become energetically favorable to move the three $n = 1$ particles with magnetic moments anti-parallel to $\vec{B}$ to one of the six unoccupied single particle states with $E = \hbar\omega(2 + \frac{3}{2})$. Finally when $2\mu B$ becomes greater than the energy separation $2\hbar\omega$ between these six states and the ground state, the anti-aligned ground state particle also moves up to an $E = \hbar\omega(2 + \frac{3}{2})$ state. Thus, the magnetic susceptibility is zero for $0 \leq \mu B \leq \hbar\omega$. It jumps to 6μ for $\hbar\omega \leq \mu B \leq 2\hbar\omega$ and finally jumps to its largest value, 8μ for $2\hbar\omega \leq \mu B$.

C. Haily Quantum solution NOV 1 2007

a.) For the singlet state we need the symmetric spatial wave function which is the product of the single particle states. The single particle states are the solutions to

$$-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\psi = E\psi$$

For the given boundary conditions this is

$$\psi_n = \sqrt{\frac{2}{a}}\cos\frac{n\pi x}{a} \qquad n = 1, 3, 5 \ldots$$

$$\psi_n = \sqrt{\frac{2}{a}}\sin\frac{n\pi x}{a} \qquad n = 2, 4, 6$$

as can be seen by substitution, normalization and parity.

The symmetric ground state is just

$$\psi_{11}^{S}(x_1, x_2) = \frac{2}{a}\cos\frac{\pi x_1}{a}\cos\frac{\pi x_2}{a} \qquad \text{Ans } 2$$

This clearly solves the 2-particle Schrodinger

$$-\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}\right)\psi_{11}^{S}(x_1, x_2) = E_{11}\psi_{11}^{S}(x_1, x_2)$$

and is properly normalized ie

$$\int_{-a/2}^{a/2}\int |\psi_{11}^{S}(x_1, x_2)|^2 dx_1 dx_2 = 1$$

This immediately gives $E_{11} = \frac{\hbar^2}{m}\left(\frac{\pi}{a}\right)^2$ Ans 2

as the ground state

2008 Quals, Hailey, Section 3, QM, Question 1?

Consider two identical, non-interacting spin ½ particles of mass m. The particles are moving in a potential given by

$V(x_1, x_2) = 0$	for x_1 and x_2 in the interval $(-a/2, a/2)$ and
$V(x_1, x_2) = \infty$	otherwise

This is just the classical 1-dimensional particle(s) in a box.

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(b) For the triplet spin state, explicitly solve the Schrödinger equation to obtain the 2-particle ground state wavefunction. Calculate the energy of the ground state.

Assume that an interaction potential between the particles is turned on. The form of the interaction is $v = v_o{*}b\ \delta(x_1 - x_2)$ where v_o is the interaction strength and b a characteristic length.

(c) Calculate the first order correction to the ground state energy in the triplet and singlet states.

(d) Provide a simple physical explanation for the magnitude of the first order correction for the singlet and triplet states.

b.) For the spin triplet we need an anti-symmetric wavefunction

$$\psi^{A}_{n_1 n_2} = \frac{1}{\sqrt{2}}\left(\psi_{n_1}(x_1)\psi_{n_2}(x_2) - \psi_{n_1}(x_2)\psi_{n_2}(x_1)\right)$$

where $n_1 \neq n_2$. The ground state wave function is

$$\psi^{A}_{12} = \frac{1}{\sqrt{2}}\frac{2}{a}\left(\cos\frac{\pi x_1}{a}\sin\frac{2\pi x_2}{a} - \sin\frac{2\pi x_1}{a}\cos\frac{\pi x_2}{a}\right)$$

This clearly solves the 2-body Schrodinger equation.

using $$-\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}\right)\psi^{A}_{12} = E_{12}\,\psi^{A}_{12}$$

we immediately obtain $E_{12} = \frac{5}{2m}\left(\frac{\hbar\pi}{a}\right)^2$ Ans 2

as the ground state.

c.) For the singlet state we need $\langle \psi^{S}_{11} | V | \psi^{S}_{11} \rangle$

$$E^{(1)} = \frac{4}{a^2}\iint \cos\frac{\pi x_1}{a}\cos\frac{\pi x_2}{a}\, V_0 b\,\delta(x_1 - x_2)\cos\frac{\pi x_1}{a}\cos\frac{\pi x_2}{a}\,dx_1\,dx_2$$

$$E^{(1)} = \frac{4}{a^2}V_0 b\int_{-a/2}^{a/2}\cos^4\frac{\pi x_1}{a}\,dx_1$$

Ans 2 $$E^{(1)} = \frac{3}{2}V_0\frac{b}{a}$$ using $\int \cos^n\theta\, d\theta = \frac{n-1}{n}\int \cos^{n-2}\theta\, d\theta$

For the triplet state the δ-fcn will yield $x_1 = x_2$ on integration over dx_2, and $\psi_{12}^{A}(x_1, x_1) = 0$

So $E^{(1)} = 0$ Ans

for triplet

d.) The antisymmetry of the spatial wavefunction in the triplet state means the e^-'s can never overlap and thus never feel the interaction potential.. In the singlet state the wavefunction allows e^- overlap, yielding non-zero $E^{(1)}$.

HEINZ QM PROBLEM – SOL'N

(a) $(H_0 + V)|\Psi\rangle = i\hbar \frac{\partial|\Psi\rangle}{\partial t}$ with $V = -\frac{e}{2}(e^{+i\omega t} + e^{-i\omega t})\mathcal{E}_0 z$

$|\Psi\rangle \equiv c_0(t)|0\rangle + c_1(t)e^{-i\omega t}|1\rangle$

where we set $E_0 = 0$ for convenience and $\hbar\omega = E_1 - E_0 = E_1$

Substituting the expansion into the Schrödinger eq:

$c_0 V|0\rangle + c_1 \hbar\omega e^{-i\omega t}|1\rangle + c_1 e^{-i\omega t} V|1\rangle =$

$i\hbar c_0'|0\rangle + i\hbar c_1' e^{-i\omega t}|1\rangle + c_1 \hbar\omega e^{-i\omega t}|1\rangle$

Now project the equation on states $\langle 0|$ and $\langle 1|$:

$c_1 e^{-i\omega t}\langle 0|V|1\rangle = i\hbar c_0'$

$c_0 \langle 1|V|0\rangle = i\hbar e^{-i\omega t} c_1'$ (*)

Here we omit terms $\langle 0|V|0\rangle$ and $\langle 1|V|1\rangle$. They vanish by parity and can also be omitted since they only contribute rapid oscillations. Keeping only terms in (*) oscillating at the same frequency:

$c_1 V_{01} = i\hbar c_0'$; $c_0 V_{10} = i\hbar c_1'$ with $2V_{10} = 2V_{01}^* = \langle 1|-ez\mathcal{E}_0|0\rangle = \mathcal{E}_0\langle 1|-ez|0\rangle \equiv \mathcal{E}_0 p_{10}$

combining the above

$c_0'' = -\Omega_R^2 c_0$ with $\Omega_R = \frac{|V_{10}|}{\hbar} = \frac{\mathcal{E}_0 p_{10}}{2\hbar}$ ||

$\therefore c_0(t) = \cos(\Omega_R t)$ for $c_0(t=0) = 1$

$P_1(t) = 1 - |c_0(t)|^2 = \sin^2(\Omega_R t)$ ||

(b) Maximum population for $\Omega_R \tau = \pi/2 \Rightarrow \mathcal{E}_0 = \frac{\pi\hbar}{e z_{10}\tau}$

$z_{10} \sim a_0 \sim 10^{-10}\,m \Rightarrow \mathcal{E}_0 \sim 2\times 10^4\,V/m$ for $\tau = 10^{-9}s$

$\therefore I = \langle S\rangle = \frac{\epsilon_0 c \mathcal{E}_0^2}{2} = 600\,kW/m^2$ $|| = \frac{P}{A} \Rightarrow A = 2\times 10^{-9}\,m^2$

Prof. Kabat

Coherent states

1. $|\lambda\rangle = \sum_{n=0}^{\infty} \frac{1}{n!} \lambda^n (\hat{a}^+)^n |0\rangle$

$\hat{a}|\lambda\rangle = \sum_{n=1}^{\infty} \frac{1}{n!} \lambda^n \hat{a} (\hat{a}^+)^n |0\rangle$

$= \sum_{n=1}^{\infty} \frac{1}{n!} \lambda^n \, n (\hat{a}^+)^{n-1} |0\rangle$

$= \lambda |\lambda\rangle$

2. $\hat{a}\psi = \lambda \psi$

$\sqrt{\frac{m\omega}{2\hbar}} \left(x + \frac{1}{m\omega} \frac{\partial}{\partial x} \right) \psi = \lambda \psi$

$\frac{1}{\psi} \frac{d\psi}{dx} = \sqrt{2\hbar m\omega}\, \lambda - m\omega x$

$\log \psi = \text{const.} + \sqrt{2\hbar m\omega}\, \lambda x - \frac{1}{2} m\omega x^2$

$\psi \sim e^{-\frac{1}{2} m\omega x^2} e^{\sqrt{2\hbar m\omega}\, \lambda x}$

3. $\langle x \rangle = \frac{\langle \lambda | \sqrt{\frac{2\hbar}{m\omega}} \frac{1}{2} (\hat{a} + \hat{a}^+) | \lambda \rangle}{\langle \lambda | \lambda \rangle} = \sqrt{\frac{2\hbar}{m\omega}} \, \mathrm{Re}\, \lambda$

$\langle p \rangle = \frac{\langle \lambda | \sqrt{2\hbar m\omega} \frac{1}{2i} (\hat{a} - \hat{a}^+) | \lambda \rangle}{\langle \lambda | \lambda \rangle} = \sqrt{2\hbar m\omega} \, \mathrm{Im}\, \lambda$

4. $|\lambda_0\rangle = \sum_{n=0}^{\infty} \frac{1}{n!} \lambda_0^n (\hat{a}^+)^n |0\rangle$

$|\lambda(t)\rangle = \sum_{n=0}^{\infty} \frac{1}{n!} \lambda_0^n e^{-i\omega(n+\frac{1}{2})t} (\hat{a}^+)^n |0\rangle$

$= e^{-i\hbar\omega t/2} \sum_{n=0}^{\infty} \frac{1}{n!} (e^{-i\omega t} \lambda_0)^n (\hat{a}^+)^n |0\rangle$

$\Rightarrow \lambda(t) = \lambda_0 e^{-i\omega t}$

Millis 07 Quantum Solution

(a) $\vec{\sigma}_1 \cdot \vec{\sigma}_2 = \frac{1}{2}\left((\vec{\sigma}_1 + \vec{\sigma}_2)^2 - \vec{\sigma}_1^2 - \vec{\sigma}_2^2\right) = \frac{1}{2}(\vec{\sigma}_1 + \vec{\sigma}_2)^2 - 3.$ Thus the spin-dependent part of the Hamiltonian is proportional to $(\vec{\sigma}_1 + \vec{\sigma}_2)$ and therefore commutes with it.

(b) The triplet states are not bound (repulsive potential). The singlet state has potential $V(r) = -3g^2/r$. The problem is thus hydrogen with $e^2 \to 3g^2$ and a mass equal to the reduced mass $\mu = m/2$. Thus eigenvalues

$$E_n = \frac{9mg^4}{2n^2}$$

(c) The bound states are fermions (because spin 1/2 and spin singlet, meaning the wave function is odd under interchange of spins; thus even under interchange of particles; thus only EVEN angular momenta are allowed.