Columbia University
Department of Physics
QUALIFYING EXAMINATION
Wednesday, January 10, 2007
9:00 AM – 11:00 AM

Modern Physics
Section 3. Quantum Mechanics

Two hours are permitted for the completion of this section of the examination. Choose 4 problems out of the 5 included in this section. (You will not earn extra credit by doing an additional problem). Apportion your time carefully.

Use separate answer booklet(s) for each question. Clearly mark on the answer booklet(s) which question you are answering (e.g., Section 3 (QM), Question 1; Section 3(QM) Question 5, etc.)

Do **NOT** write your name on your answer booklets. Instead clearly indicate your **Exam Letter Code**

You may refer to the single handwritten note sheet on 8 ½ x 11" paper (double-sided) you have prepared on Modern Physics. The note sheet cannot leave the exam room once the exam has begun. This note sheet must be handed in at the end of today's exam. Please include your Exam Letter Code on your note sheet. No other extraneous papers or books are permitted.

Simple calculators are permitted. However, the use of calculators for storing and/or recovering formulae or constants is NOT permitted.

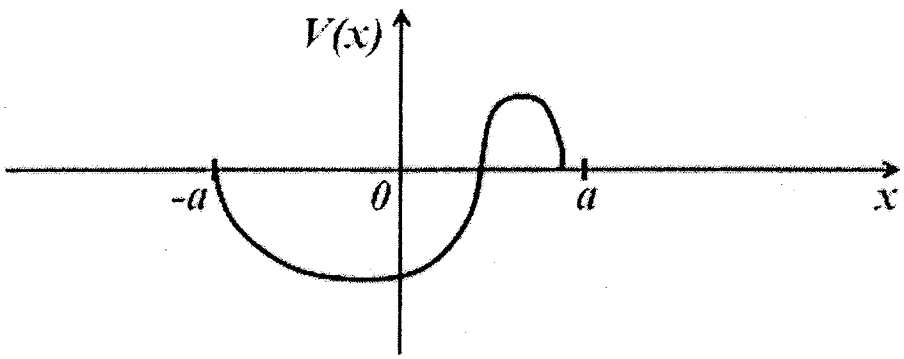
Questions should be directed to the proctor.

Good luck!!

1. Consider a particle moving in *one* dimension under the influence of a potential $\lambda V(x)$, with $V(x) \to 0$ as $x \to \infty$. The Hamiltonian is

$$H = \frac{p^2}{2m} + \lambda V(x)$$

For simplicity, assume that the potential is zero in the region $|x| > a$, as shown in the figure.



(a) Use the variational principle to show that as long as $\int V(x)dx < 0$ this Hamiltonian has a bound state for an arbitrarily small but positive coupling λ.

(b) Give an upper bound for the energy of this bound state for $\lambda << 1$.

(c) What does this approach yield for bound states in three dimensions?

2. The deuteron is a bound state of the neutron and proton. Consider a simple model of the deuteron in which the nuclear potential is a spin-dependent 3-dimensional, radial square well.

$$V(r) = \frac{V_c(r)}{2}\left(1 + \hat{\sigma}_n \cdot \hat{\sigma}_p\right) \quad \text{where}$$

$$V_c(r) = -V_0 \quad 0 < r \le a, \qquad V_0 > 0$$

$$= 0 \qquad r > a$$

$\hat{\sigma}_n$ and $\hat{\sigma}_p$ are the Pauli spin matrices for the neutron and proton, respectively.

(a) What are the possible values of the total spin of the deuteron?

(b) Construct explicit expressions for the total spin eigenstates in terms of the neutron and proton spinors.

(c) Explicitly evaluate $V(r)$ for the possible deuteron spin states. Which spin state leads to bound states of the deuteron, and which to unbound states?

(d) The ground state of the deuteron has relative angular momentum $l = 0$. Solve the Schrödinger equation to obtain the ground state wavefunctions. You need not normalize the wavefunctions.

3. Consider a two-level quantum system described by Hamiltonian H, with states $|a\rangle$ and $|b\rangle$ and energies $E_a = 0$ and $E_b = E_0$. The system is initially in state $|a\rangle$. Suppose that a constant perturbation H' is applied from time $t = 0$ until some arbitrary subsequent time t.

Find the probability $P_b(t)$ of being in state $|b\rangle$ just after the perturbation has been removed. Sketch the variation with time t, indicating characteristic time scales on the plot. (Partial credit will be given for an accurate plot, even if the information is incomplete.) The problem should be solved *without approximation* for two forms of the perturbation H'.

(a) $\langle a|H'|a\rangle = U_a$, $\langle b|H'|b\rangle = U_b$, $\langle a|H'|b\rangle = \langle b|H'|a\rangle = 0$

(b) $\langle a|H'|a\rangle = 0$, $\langle b|H'|b\rangle = 0$, $\langle a|H'|b\rangle = \langle b|H'|a\rangle = U$.

4. Consider a particle of mass m moving in one dimension and subject to the potential

$$V(x) = \infty \quad x < 0$$

$$V(x) = V_0 a\delta(x-a) \quad x > 0$$

shown in the sketch.

For $x > a$ and $E > 0$ the wave function may be written

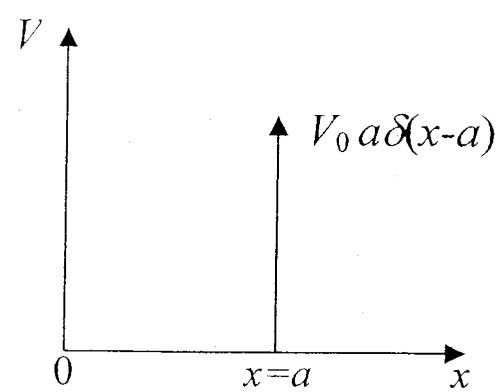
$$\psi(x) = Ae^{-ikx} + Be^{ikx+i\phi(k)}$$

with A, B, $\phi(k)$ real and A, $B > 0$.

(a) Conservation of current implies a relation between $|A|^2$ and $|B|^2$. Please state this relation.

(b) (No calculation required). Information about the scattering is contained in the 'phase shift' $\phi(k)$ which depends on the wavenumber k. Please state the limiting behavior of ϕ as $|k|$ becomes large and as $|k| \to 0$ and give the physical reason for your answer.

(c) Determine $\phi(k)$ as a function of V_0, a, m and k.



5. A localized electron ($s = ½$) is subjected to a time dependent magnetic field $\boldsymbol{B}(t)$. The corresponding Hamiltonian has the form

$$\hat{H}(t) = g_L \mu_B \hat{\boldsymbol{s}} \cdot \boldsymbol{B}(t)$$

where μ_B is the Bohr magneton, g_L is the g-factor, and $\hat{\boldsymbol{s}}$ is the operator for electron spin. At $t = 0$ the wavefunction in the basis $s_z = \pm ½$ has the form

$$\psi(t=0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

(a) Find $\psi(t)$ for $\boldsymbol{B} = (0,\, 0,\, B_z(t))$, where $B_z(t)$ is an arbitrary function.

(b) Find $\psi(t)$ for $\boldsymbol{B} = (B_\perp \cos\omega t,\, B_\perp \sin\omega t,\, B_z)$, where B_z and $B_\perp$ do not depend on time.

(c) Find $\psi(t)$ for $\boldsymbol{B}(t)$ to be an arbitrary slow function, i.e.

$$\hbar |d\boldsymbol{B}/dt| << g_L \mu_B \boldsymbol{B}^2$$

ALTSCHULER Sec 3 # 1

Solution of the problem on Quantum Mechanics

1. **Use the variation principle to show that as long as $\int V(x)dx < 0$ this Hamiltonian has a bound state for arbitrarily weak but positive coupling λ. (5 points)**

According to the variation principle

$$\frac{\langle\psi|H|\psi\rangle}{\langle\psi|\psi\rangle} \geq E_0,$$

where E_0 is the ground state energy, is the Hamiltonian, and ψ is any normalizable function with arguments appropriate for the unknown wave function of the system. In our case

$$H = \frac{p^2}{2m} + \lambda V(x) \qquad p = -i\hbar\frac{\partial}{\partial x}$$

$$\langle\psi|H|\psi\rangle = \left\langle\psi\left|\frac{p^2}{2m}\right|\psi\right\rangle + \lambda\int_{-\infty}^{\infty}|\psi(x)|^2 V(x)dx$$

If $\psi(x)$ is real and $\psi(\pm\infty) = 0$

$$\left\langle\psi\left|\frac{p^2}{2m}\right|\psi\right\rangle = -\frac{\hbar^2}{2m}\int_{-\infty}^{\infty}\psi(x)\frac{d^2\psi(x)}{dx^2}dx = \frac{\hbar^2}{2m}\int_{-\infty}^{\infty}\left[\frac{d\psi(x)}{dx}\right]^2 dx$$

Let us use, e.g., the following probe function: $\psi(x) = e^{-|x|/b}$:

$$\left\langle\psi\left|\frac{p^2}{2m}\right|\psi\right\rangle = \frac{\hbar^2}{b^2 m}\int_0^{\infty}e^{-2x/b}dx = \frac{\hbar^2}{2bm}$$

Provided that $b >> a$ one can neglect dependence of $\psi(x)$ on x in the potential term:

$$\lambda\int_{-\infty}^{\infty}|\psi(x)|^2 V(x)dx \approx \lambda|\psi(0)|^2\int_{-\infty}^{\infty}V(x)dx = \lambda\int_{-\infty}^{\infty}V(x)dx$$

Finally, for the normalization factor we have

$$\langle\psi|\psi\rangle = 2\int_0^{\infty}e^{-2x/b}dx = b$$

We now can write the variation principle as

$$E_0 \leq \frac{\langle\psi|H|\psi\rangle}{\langle\psi|\psi\rangle} = \frac{\hbar^2}{2mb^2} + \frac{\lambda}{b}\int_{-\infty}^{\infty}V(x)dx$$

Now one can see that for large enough value of b, i.e. for large enough size of the probe function the left hand side of this inequality is negative if $\lambda\int_{-\infty}^{\infty}V(x)dx < 0$. We thus conclude that $E_0 < 0$

2. **Give an upper bound for the energy of this bound state for $\lambda << 1$ (5 points)**

One can use the inequality

$$E_0 \le \frac{\langle\psi|H|\psi\rangle}{\langle\psi|\psi\rangle} = \frac{\hbar^2}{2mb^2} + \frac{\lambda}{b}\int_{-\infty}^{\infty} V(x)dx \equiv F(b^{-1})$$

to estimate the ground state energy. Let us minimize $F(b^{-1})$. Evaluating the derivative

$$\frac{dF}{d(b^{-1})} = \frac{\hbar^2}{mb} + \lambda\int_{-\infty}^{\infty} V(x)dx$$

we find that it vanishes at

$$b = -\frac{\hbar^2}{m\lambda\int_{-\infty}^{\infty} V(x)dx}$$

($\lambda\int_{-\infty}^{\infty} V(x)dx < 0$!)

Note that for $\lambda \to 0$ $\quad b \to \infty$ and our condition $a<<b$ is satisfied. Now we can determine the minimal value of the function $F(b^{-1})$ and obtain an estimation for the ground state energy

$$E_0 \le F(b^{-1})_{\min} = -\frac{m}{2\hbar^2}\left(\lambda\int_{-\infty}^{\infty} V(x)dx\right)^2$$

3. **Would the same approach work in three dimensions? Explain why. (5 points)**

The answer is NO. In three dimensions the potential term $\lambda\int|\psi(\vec{r})|^2 V(\vec{x})d^3r$ will be proportional to the *third power* of the inverse size of the probe function b^{-1}, while the kinetic term $\left\langle\psi\left|\frac{p^2}{2m}\right|\psi\right\rangle \Big/ \langle\psi|\psi\rangle$ will still behave like b^{-2}. As a result the sum of these two terms will be positive for large b, in contract with one-dimensional case, when this sum turned out to be negative for $\lambda\int_{-\infty}^{\infty} V(x)dx < 0$ and $b \to \infty$. Only strong enough potential can host a bound state in three dimensions.

Quantum Hailey   sec 3 #2

NOV 28 2006

The deuteron is a bound state of the neutron and proton. Consider a simple model of the deuteron in which the nuclear potential is a spin-dependent 3-dimensional, radial square well

$$V(r) = \frac{V_c(r)}{2}\left(1 + \hat{\sigma}_n \cdot \hat{\sigma}_p\right) \quad \text{where}$$

$$\begin{aligned} V_c(r) &= -V_0 \qquad 0 < r \leq a \qquad V_0 > 0 \\ &= 0 \qquad\quad r > a \end{aligned}$$

$\hat{\sigma}_n$ and $\hat{\sigma}_p$ are the Pauli spin matrices for the neutron and proton respectively.

a.) What are the possible values of the total spin of the deuteron?

b.) Construct explicit expressions for the total spin eigenstates in terms of the neutron and proton spinors.

c.) Explicitly evaluate $V(r)$ for the possible deuteron spin states. Which spin state leads to bound states of the deuteron, and which to unbound states?

d.) The ground state of the deuteron has relative angular momentum $l = 0$. 1

Solve the Schrodinger equation to obtain the ground state wavefunctions. You need not normalize the wavefunctions.

d.) Using your eigenvalue equation from (c.), and assume the ground state energy ε satisfies $|\varepsilon| << V_0$. Obtain an algebraic expression for V_0. Use the value $a \simeq 1.5$ fm obtained from scattering studies to explicitly evaluate V_0, the depth of the nuclear potential.

Solution : Quantum Hailey

a.) $S = 0, 1$

b.) $|1\,1\rangle = |\tfrac{1}{2}\,\tfrac{1}{2}\rangle_p\,|\tfrac{1}{2}\,\tfrac{1}{2}\rangle_n$

$|1\,0\rangle = \left[|\tfrac{1}{2}\,-\tfrac{1}{2}\rangle_p\,|\tfrac{1}{2}\,\tfrac{1}{2}\rangle_n + |\tfrac{1}{2}\,\tfrac{1}{2}\rangle_p\,|\tfrac{1}{2}\,-\tfrac{1}{2}\rangle_n\right]\frac{1}{\sqrt{2}}$

$|1\,-1\rangle = |\tfrac{1}{2}\,-\tfrac{1}{2}\rangle_p\,|\tfrac{1}{2}\,-\tfrac{1}{2}\rangle_n$

$|0\,0\rangle = \frac{1}{\sqrt{2}}\left[|\tfrac{1}{2}\,-\tfrac{1}{2}\rangle_p\,|\tfrac{1}{2}\,\tfrac{1}{2}\rangle_n - |\tfrac{1}{2}\,\tfrac{1}{2}\rangle_p\,|\tfrac{1}{2}\,-\tfrac{1}{2}\rangle_n\right]$

c.) $\hat{S} = \hbar/2\,\hat{\sigma} \qquad \hat{S}_n \cdot \hat{S}_p = \frac{\hbar^2}{4}\,\hat{\sigma}_n \cdot \hat{\sigma}_p$

$\hat{S} = \hat{S}_p + \hat{S}_n \;; \quad \hat{S}_n \cdot \hat{S}_p = \frac{1}{2}\left(\hat{S}^2 - \hat{S}_n^2 - \hat{S}_p^2\right)$

$\hat{\sigma}_n \cdot \hat{\sigma}_p = \frac{2}{\hbar^2}\left(\hat{S}^2 - \hat{S}_n^2 - \hat{S}_p^2\right)$

$\hat{\sigma}_n \cdot \hat{\sigma}_p = 2\left[S(S+1) - \tfrac{1}{2}(\tfrac{1}{2}+1) - \tfrac{1}{2}(\tfrac{1}{2}+1)\right]$

$\hat{\sigma}_n \cdot \hat{\sigma}_p = 2\left[S(S+1) - \frac{3}{2}\right]$

when $\hat{\sigma}_n \cdot \hat{\sigma}_p$ operates on a state of the deuteron with $S=1 \Rightarrow$

$$\hat{\sigma}_n \cdot \hat{\sigma}_p |1\rangle = +1 |1\rangle$$

For $S=0$ $\quad \hat{\sigma}_n \cdot \hat{\sigma}_p |0\rangle = -3 |0\rangle$

so $\quad V(r) = V_c(r) \qquad S=1$

$$V(r) = -V_c(r) \qquad S=0$$

For the given form of $V_c(r)$, the $S=1$ state is bound ie

$$V_c(r) = -V_0 < 0 .$$

d.) Starting from the 3-d Schrodinger eqn.

$$-\frac{\hbar^2}{2m}\frac{1}{r^2}\frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r}\psi + \frac{\hat{L}^2}{2mr^2}\psi + V(r)\psi = E\psi$$

use the substitution $\psi(r) = \frac{\chi(r)}{r}Y_{lm}$ or just remember that this gives

$$-\frac{\hbar^2}{2m}\frac{d^2}{dr^2}\chi + \frac{l(l+1)\hbar^2}{2mr^2}\chi + V(r)\chi = E\chi$$

where m is the reduced mass of the deuteron. For $l=0$ this is just the 1-d Schrodinger equation

$$-\frac{\hbar^2}{2m}\frac{d^2}{dr^2}\chi + V\chi = E\chi$$

with The $S=1$ bound state solution for $V(r)$ ie

$$V(r) = -V_0 \qquad 0 < r \le a$$
$$= 0 \qquad r > a$$

$$-\frac{\hbar^2}{2m}\frac{d^2}{dr^2}\chi - V_0\chi = \varepsilon\chi \qquad 0 < r \le a$$

$$-\frac{\hbar^2}{2m}\frac{d^2}{dr^2}\chi = \varepsilon\chi \qquad r > a$$

let $\alpha^2 = -\frac{2m\varepsilon}{\hbar^2}$; $K^2 = \frac{2m}{\hbar^2}(\varepsilon + V_0)$

Then $\frac{d^2}{dr^2}\chi - \alpha^2\chi = 0 \qquad r > a$

$$\frac{d^2}{dr^2}\chi + K^2\chi = 0 \qquad 0 < r \le a$$

Note $K^2 = \frac{2m}{\hbar^2}(\varepsilon + V_0) = \frac{2m}{\hbar^2}(-|\varepsilon| + V_0) \approx \frac{2m V_0}{\hbar^2}$

by assumption of problem.

The obvious solutions are

$$\left.\begin{aligned} \chi &= A\sin Kr \qquad 0 < r \le a \\ \chi &= B e^{-\alpha r} \qquad r > a \end{aligned}\right\} \text{Answer}$$

since $\psi \sim \frac{\chi}{r}$ must be finite at origin

And $\chi \to 0$ as $r \to \infty$.

d.) $\chi = A \sin kr \qquad 0 < r \leq a$

$\chi = B e^{-\alpha r} \qquad r > a$

χ, χ' continuous imply

$$A \sin ka = B e^{-\alpha a}$$

$$kA \cos ka = -\alpha B e^{-\alpha a}$$

Dividing $\quad k \cot ka = -\alpha$

$$\cot ka = -\frac{\alpha}{k} = -\frac{\sqrt{\frac{-2m\varepsilon}{\hbar^2}}}{\sqrt{\frac{2m}{\hbar^2} V_0}} = -\sqrt{\frac{-\varepsilon}{V_0}}$$

$\cot ka \simeq 0$ for $\frac{-\varepsilon}{V_0} \ll 1$

$$ka \simeq \pi/2 \qquad k^2 = \frac{2m}{\hbar^2} V_0 \simeq \frac{\pi^2}{4a^2}$$

$$V_0 \simeq \frac{\pi^2 \hbar^2}{8 m a^2} \quad \underline{\text{Ans}} \qquad m \simeq m_p/2$$

$$V_0 \simeq \frac{\pi^2 \hbar^2}{4 m_p a^2} = \frac{\pi^2 (\hbar c)^2}{4 m_p c^2 a^2} = \frac{\pi^2 (197 \text{MeV-fm})^2}{4 \times 939 \text{MeV} \times (1.5 \text{fm})^2}$$

$V_0 \simeq 45$ MeV. $\quad \underline{\text{Ans}}$

QM Problem (Heinz)

(a) For a diagonal perturbation, the system remains in the same state $\Rightarrow P_b(t) = 0$ ||

(b) For $H = H_0 + H' = \begin{pmatrix} 0 & 0 \\ 0 & E_0 \end{pmatrix} + \begin{pmatrix} 0 & u \\ u & 0 \end{pmatrix} = \begin{pmatrix} 0 & u \\ u & E_0 \end{pmatrix}$

we have new eigenstates and eigenvalues

$$E_{A,B} = \frac{1}{2}\left(E_0 \mp \sqrt{E_0^2 + 4u^2}\right) \equiv \frac{1}{2}(E_0 \mp E')$$

$$\begin{pmatrix} |A\rangle \\ |B\rangle \end{pmatrix} = \begin{pmatrix} u/\sqrt{E_A^2+u^2} & E_A/\sqrt{E_A^2+u^2} \\ u/\sqrt{E_B^2+u^2} & E_B/\sqrt{E_B^2+u^2} \end{pmatrix} \begin{pmatrix} |a\rangle \\ |b\rangle \end{pmatrix}$$

The evolution of State $|a\rangle$ from $t=0$ under H is

$|\psi(t)\rangle = e^{-iE_At/\hbar}|A\rangle\langle A|a\rangle + e^{-iE_Bt/\hbar}|B\rangle\langle B|a\rangle$, so

$$P_b(t) = |\langle b|\psi(t)\rangle|^2 = |\langle b|A\rangle\langle A|a\rangle e^{-iE_At/\hbar} + \langle b|B\rangle\langle B|a\rangle e^{-iE_Bt/\hbar}|^2$$

$$= |\langle b|A\rangle\langle A|a\rangle|^2 |e^{iE't/2\hbar} - e^{-iE't/2\hbar}|^2$$

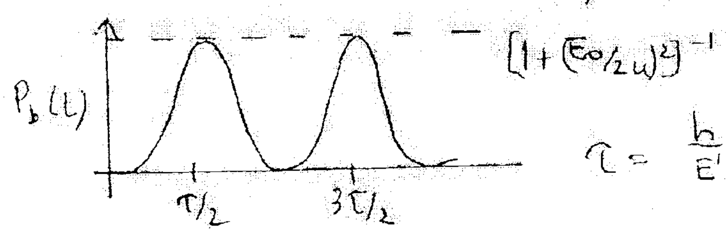
Since $P_b(0) = 0 \Rightarrow \langle b|A\rangle\langle A|a\rangle = -\langle b|B\rangle\langle B|a\rangle$ and the common phase factor $e^{-iE_0t/2\hbar}$ can be eliminated.

$$\therefore\ P_b(t) = 4|\langle b|A\rangle\langle A|a\rangle|^2 \sin^2(E't/2\hbar)$$

$$4|\langle b|A\rangle\langle A|a\rangle|^2 = 4E_A u^2/(E_A^2+u^2)^2 \quad \text{from } (*)$$

$$= 1/[1+(E_0/2u)^2] \quad \text{after some algebra}$$

$$\| \; P_b(t) = \left[1+\left(\frac{E_0}{2u}\right)^2\right]^{-1} \sin^2(E't/2\hbar) \quad \text{with } E' = (E_0^2+4u^2)^{1/2}$$



$$\tau = \frac{h}{E'} = \frac{h}{\sqrt{E_0^2+4u^2}}$$

Note: The main features of this result should be apparent without detailed analysis, just from the concept of mode beating. Knowledge of the new eigenvalues yields the full plot of $P_b(t)$ except for the coefficient.

Sec 3 #5

Solutions:

$$i\hbar\partial_t\psi = g_L\mu_B \boldsymbol{B}(t)\cdot\hat{s}\psi = \frac{1}{2}g_L\mu_B \boldsymbol{B}(t)\cdot\hat{\sigma}\psi \tag{1}$$

where $\hat{\sigma}_{x,y,z}$ are the Pauli matrices. [1pt]

a) Find $\psi(t)$ for $\boldsymbol{B} = (0;\ 0;\ B_z(t))$, where $B_z(t)$ is an arbitrary function.

$$\psi(t) = \exp\left[-\frac{i}{2\hbar}g_L\mu_B\int_0^t dt_1 B_z(t_1)\hat{\sigma}_z\right]\psi(0) = \begin{pmatrix}\exp\left[-\frac{i}{2\hbar}g_L\mu_B\int_0^t dt_1 B_z(t_1)\right]\\ 0\end{pmatrix} \tag{2}$$

[4pts].

b) Find $\psi(t)$ for $\boldsymbol{B} = (B_\perp\cos\omega t;\ B_\perp\sin\omega t B_\perp;\ B_z)$.

Looking for for the solution in the form

$$\psi(t) = \exp\left[\frac{i\omega t\hat{\sigma}_z}{2}\right]\tilde{\psi}(t) \tag{3}$$

we obtain from Eq. (1)

$$i\hbar\partial_t\tilde{\psi} = \left[\frac{1}{2}g_L\mu_B \boldsymbol{B}_\perp\hat{\sigma}_x + \left(\frac{\hbar\omega}{2} + \frac{1}{2}g_L\mu_B B_z\right)\hat{\sigma}_z\right]\tilde{\psi} \tag{4}$$

which has the time independent Hamiltonian. Solving Eq. (4), we find

$$\tilde{\psi}(t) = \exp\left\{-it\left[\frac{1}{2\hbar}g_L\mu_B \boldsymbol{B}_\perp\hat{\sigma}_x + \left(\frac{\omega}{2} + \frac{1}{2\hbar}g_L\mu_B B_z\right)\hat{\sigma}_z\right]\right\}\tilde{\psi}(0); \tag{5}$$

Finally, with the help of Eq. (3), one obtains

$$\psi(t) = \begin{pmatrix} e^{i\omega t/2}\left\{\cos\frac{\Omega t}{2} - i\left[\frac{\omega + g_L\mu_B B_z/\hbar}{\Omega}\right]\sin\frac{\Omega t}{2}\right\}\\ -ie^{-i\omega t/2}\left[\frac{g_L\mu_B B_\perp/\hbar}{\Omega}\right]\sin\frac{\Omega t}{2}\end{pmatrix}, \tag{6}$$

where

$$\Omega = \left[\left(\omega + \frac{1}{\hbar}g_L\mu_B B_z\right)^2 + \left(\frac{1}{\hbar}g_L\mu_B B_\perp\right)^2\right]^{1/2}.$$

[6pts]

Sec 3 #5

c) Find $\psi(t)$ for

$$\boldsymbol{B}(t) = \begin{cases} (0;\ 0;\ B_z); & 0 < t < t_1; \\ (B_x;\ 0;\ 0); & t_1 < t < t_2; \\ (0;\ 0;\ B_z); & t > t_2; \end{cases}$$

where $t_2 > t_1$, and $B_{x,z}$ do not depend on time.

$$\psi(t) = \begin{cases} \exp\left[-\frac{it}{2\hbar} g_L \mu_B B_z \sigma_z\right] \psi(0) = \begin{pmatrix} \exp\left[-\frac{it}{2\hbar} g_L \mu_B B_z\right] \\ 0 \end{pmatrix}; & 0 < t < t_1; \\ \exp\left[-\dfrac{i(t-t_1)}{2\hbar} g_L \mu_B B_x \sigma_x\right] \exp\left[-\dfrac{it_1}{2\hbar} g_L \mu_B B_z \sigma_z\right] \psi(0) & \\ \quad = \begin{pmatrix} \exp\left[-\frac{it_1}{2\hbar} g_L \mu_B B_z\right] \cos \frac{(t-t_1) g_L \mu_B B_x}{2\hbar} \\ -i \sin \frac{(t-t_1) g_L \mu_B B_x}{2\hbar} \exp\left[-\frac{it_1}{2\hbar} g_L \mu_B B_z\right] \end{pmatrix}; & t_1 < t < t_2; \\ \exp\left[-\dfrac{i(t-t_2}{2\hbar} g_L \mu_B B_z \sigma_z\right] \exp\left[-\dfrac{i(t_2-t_1)}{2\hbar} g_L \mu_B B_x \sigma_x\right] & \\ \qquad \times \exp\left[-\dfrac{it_1}{2\hbar} g_L \mu_B B_z \sigma_z\right] \psi(0); & t > t_2; \\ \quad = \begin{pmatrix} \exp\left[-\frac{i(t-t_2+t_1)}{2\hbar} g_L \mu_B B_z\right] \cos \frac{(t_2-t_1) g_L \mu_B B_x}{2\hbar} \\ -i \sin \frac{(t_2-t_1) g_L \mu_B B_x}{2\hbar} \exp\left[\frac{i(t-t_2-t_1)}{2\hbar} g_L \mu_B B_z\right] \end{pmatrix} & \end{cases} \tag{7}$$

[4pts]

d) Find $\psi(t)$ for $\boldsymbol{B}(t)$ to be an arbitrary slow function, i.e.

$$\hbar |d\boldsymbol{B}/dt| \ll g_L \mu_B \boldsymbol{B}^2. \tag{8}$$

Let us represent $\boldsymbol{B}(t)$ as

$$\boldsymbol{B(t)} = |B(t)| \left[\sin\theta(t)\cos\phi(t);\ \sin\theta(t)\sin\phi(t);\ \cos\theta(t)\right]. \tag{9}$$

Then

$$\boldsymbol{B}(t)\cdot\hat{\boldsymbol{\sigma}} = |B(t)| \hat{U}(t) \hat{\sigma}_z \hat{U}^\dagger(t); \quad \hat{U}^\dagger \hat{U} = \mathbb{1}; \quad \hat{U} = \exp[i\phi(t)\hat{\sigma}_z/2] \exp[-i\theta(t)\hat{\sigma}_y/2] exp[-i\phi(t)\hat{\sigma}_z/2]. \tag{10}$$

Looking for the solution in the form

$$\psi(t) = \hat{U}(t)\tilde{\psi}(t); \quad \tilde{\psi}(0) = \hat{U}^\dagger(0)\psi(0); \tag{11}$$

we obtain from Eq. (4)

$$i\hbar\partial_t\tilde{\psi}(t)\left[\frac{g_L\mu_B|B(t)|\sigma_z}{2}-i\hbar\hat{U}^\dagger\partial_t\hat{U}\right]\tilde{\psi}. \tag{12}$$

Because of the condition (8), one can replace

$$\hat{U}^\dagger\partial_t\hat{U}=\hat{\sigma}_z\mathrm{Tr}\frac{\hat{\sigma}_z}{2}\hat{U}^\dagger\partial_t\hat{U} \tag{13}$$

and obtain from Eqs. (12) and (11)

$$\psi(t)=\hat{U}(t)\exp[-i\sigma_z\left(\varphi_D(t)+\varphi_B(t)\right)]\hat{U}^\dagger(0)\psi(0), \tag{14}$$

where

$$\varphi_D(t)=-\int_0^t dt_1\frac{g_L\mu_B|B(t_1)|}{2\hbar};\quad \varphi_B(t)=-i\int_0^t dt_1\mathrm{Tr}\frac{\hat{\sigma}_z}{2}\hat{U}^\dagger\partial_t\hat{U}=\frac{1}{2}\int_0^t dt_1\dot{\phi}(t_1)\left[1-\cos\theta(t_1)\right]$$

are the dynamical and Berry phase respectively. [9pts for dynamical phase; 6pts for Berry phase.]

Columbia University
Department of Physics
QUALIFYING EXAMINATION
Wednesday, January 10, 2007
11:10 AM – 1:10 PM

Modern Physics
Section 4. Relativity and Applied Quantum Mechanics

Two hours are permitted for the completion of this section of the examination. Choose 4 problems out of the 5 included in this section. (You will not earn extra credit by doing an additional problem). Apportion your time carefully.

Use separate answer booklet(s) for each question. Clearly mark on the answer booklet(s) which question you are answering (e.g., Section 4 (Relativity and Applied QM), Question 2; Section 4(Relativity and Applied QM) Question 3, etc.)

Do **NOT** write your name on your answer booklets. Instead clearly indicate your **Exam Letter Code**

You may refer to the single handwritten note sheet on 8 ½ x 11" paper (double-sided) you have prepared on Modern Physics. The note sheet cannot leave the exam room once the exam has begun. This note sheet must be handed in at the end of today's exam. Please include your Exam Letter Code on your note sheet. No other extraneous papers or books are permitted.

Simple calculators are permitted. However, the use of calculators for storing and/or recovering formulae or constants is NOT permitted.

Questions should be directed to the proctor.

Good luck!!

1. A proton with $\gamma = \sqrt{1-\left(v^2/c^2\right)}$ collides elastically with a proton at rest. If the two protons rebound with equal energies, what is the angle θ between them?

2. Consider a charged pion decaying at rest via $\pi^+ \to \mu^+ \nu_\mu$. Assume the ν_μ has a very small, but non-zero mass. Show that the magnitude of the 3-momentum, p, of the muon is reduced, compared to the case with a *massless* neutrino, by a factor

$$\frac{\Delta p}{p} = -\frac{m_\nu^2 (m_\pi^2 + m_\mu^2)}{(m_\pi^2 - m_\mu^2)^2}$$

where m_π, m_μ and m_ν are the pion, muon and neutrino rest masses, respectively.

3. When we calculate the energy spectrum and wavefunctions of an atom we do it in an isotropic environment, in which we artificially define an x,y,z coordinate system.

For an atom in a solid, or an impurity atom in a solid, the environment is no longer isotropic, but has a particular, lower symmetry. Under these circumstances it is often a good approach to construct new wavefunctions, having the correct symmetry of the crystalline environment, from linear combinations of wavefunctions of the isotropic case. The following problems require you to find such linear combinations for the two most common cases.

(a) A set of normalized and mutually orthogonal p-state wavefunctions for an atom can be written in the form:

$$p_x = x\,f(r)\ ,\ p_y = y\,f(r)\ ,\ p_z = z\,f(r)$$

Consider the linear combination $\psi = a_x p_x + a_y p_y + a_z p_z$.

Find four sets of coefficients (a_x, a_y, a_z) that give the normalized p-state wavefunctions with positive lobes pointing towards the corners of a regular tetrahedron. (Remember that four of the corners of a cube are corners of an inscribed tetrahedron.)

(b) Consider the linear combination: $\varphi = bs + c\psi$, where ψ is one of the four wavefunctions calculated above and s is an s-state wave function, normalized and orthogonal to p_x, p_y and p_z. Find values of b and c, which make the four resulting φ wavefunctions orthogonal to each other and normalized. Write out these four wavefunctions in terms of p_x, p_y, p_z and s. (These are the **sp^3** hybrid wavefunctions.)

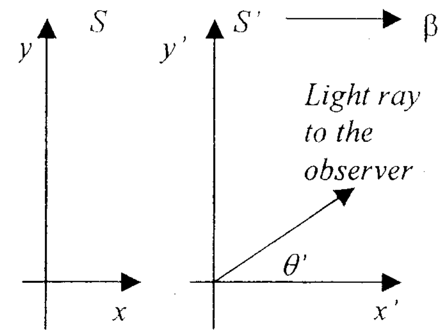
(c) The **sp^2** hybrid wavefunctions, which are involved in the bonding of a two-dimensional layer of carbon in graphite, are of the form:

$$\xi = \alpha s + \beta p_x + \gamma p_y$$

where s, p_x, and p_y are s and p wavefunctions as defined in the previous questions, (a) and (b).

Find values for α, β and γ that give three normalized mutually orthogonal wavefunctions with positive lobes directed at 120° with respect to each other in the x-y plane.

4. Consider the S and S' frames as shown. A ray of light of frequency ν' is emitted from the origin of the S' frame at an angle θ' towards an observer who is at rest in S.



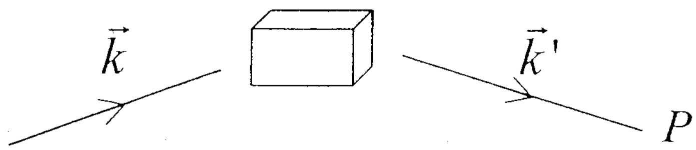
(a) Derive an expression for the frequency of the light as observed in S, i.e. find ν in terms of ν', β, θ'.

(b) Under what conditions is there a Doppler effect observed that is a purely relativistic effect?

5. Suppose a crystal's structure is such that $\vec{a}_1$, $\vec{a}_2$ and $\vec{a}_3$ define the distance between atoms in directions x_1, x_2 and x_3. The crystal has N_1, N_2 and N_3 atoms in directions x_1, x_2 and x_3 respectively so that the position of an atom in the crystal can be written as

$$\vec{r}_\alpha = n_1\vec{a}_1 + n_2\vec{a}_2 + n_3\vec{a}_3$$

where n_1, n_2 and n_3 are integers. Incident on the crystal is a plane wave $\psi \sim e^{i(\vec{k}\cdot\vec{r}-\omega t)}$. Assuming that the incoming wave is scattered elastically by the atoms in the crystal, what is the intensity of the scattered wave at a distant point P?



Hints: $\sum_{j=0}^{N-1} e^{j\alpha} = \frac{1-e^{N\alpha}}{1-e^{\alpha}}$

Elastic scattering implies that $|\vec{k}| = |\vec{k}'|$

2007 Quals Problem SOLUTION from Emlyn Hughes
Section 4 - Relativity
11/22/06

Proton-Proton Collision

Since the energies of the protons after the collision are equal, they will rebound at the same angle $\theta/2$ relative to the initial momentum of the proton (see Figure S.2.17). Again we use $c = 1$. In these units before the

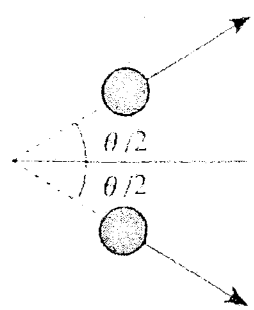


Figure **S.2.17**

collision

$$p = m\beta\gamma$$

$$E = m\gamma$$

Using momentum conservation, we have

$$m\beta\gamma = 2m\tilde{\beta}\tilde{\gamma}\cos\left(\frac{\theta}{2}\right) \qquad \text{(S.2.17.1)}$$

or

$$\beta\gamma = 2\tilde{\beta}\tilde{\gamma}\cos\frac{\theta}{2} \qquad \text{(S.2.17.2)}$$

where $\tilde{\beta}$ and $\tilde{\gamma}$ stand for β and γ after the collision. Energy conservation yields

$$m + m\gamma = 2m\tilde{\gamma} \qquad \text{(S.2.17.3)}$$

or

$$\gamma + 1 = 2\tilde{\gamma} \qquad \text{(S.2.17.4)}$$

Now,

$$\beta\gamma = \sqrt{\gamma^2 - 1} \qquad \text{(S.2.17.5)}$$

So, we obtain by using (S.2.17.4)

$$\tilde{\beta}\tilde{\gamma} = \sqrt{\tilde{\gamma}^2 - 1} = \sqrt{(\gamma+1)^2/4 - 1} \qquad \text{(S.2.17.6)}$$

Substituting (S.2.17.6) into (S.2.17.2) and using (S.2.17.5) gives

$$\sqrt{\gamma^2 - 1} = 2\sqrt{(\gamma+1)^2/4 - 1}\ \cos\frac{\theta}{2}$$

$$\gamma^2 - 1 = (\gamma^2 + 2\gamma - 3)\cos^2\frac{\theta}{2} = (\gamma - 1)(\gamma + 3)\cos^2\frac{\theta}{2}$$

$$\cos\theta = 2\cos^2\frac{\theta}{2} - 1 = \frac{\gamma - 1}{\gamma + 3}$$

For $\gamma \approx 1$, i.e., in the classical limit of low velocity, $\cos\theta = 0$, and we obtain the familiar result that the angle between billiard balls rebounding with equal energy is $90°$. If $\gamma \gg 1$ (extremely relativistic case), then $\cos\theta \approx 1$ and $\theta \to 0$.

Neglecting m_ν^4 terms can solve ⊛ for p^2 to get

$$\circledast\!\!\circledast \quad p^2 = \frac{(m_\pi^2 - m_\mu^2)^2 + 2m_\nu^2(m_\pi^2 - m_\mu^2) - 4m_\pi^2 m_\nu^2}{4m_\pi^2}$$

Now consider 2 cases:

(i) $p \equiv p_0$ when $m_\nu = 0$

$$\circledast\!\!\circledast \Rightarrow p_0^2 = \frac{(m_\pi^2 - m_\mu^2)^2}{4m_\pi^2}$$

(ii) $p \equiv p_m$ when $m_\nu \equiv m \neq 0$

$$\circledast\!\!\circledast \Rightarrow p_m^2 = \frac{(m_\pi^2 - m_\mu^2)^2 - 2m_\nu^2(m_\pi^2 + m_\mu^2)}{4m_\pi^2}$$

So $$p_m^2 - p_0^2 = \frac{-2m_\nu^2(m_\pi^2 + m_\mu^2)}{4m_\pi^2}$$

$p_m \approx p_0$ for m_ν small

But $$p_m^2 - p_0^2 = \underbrace{(p_m - p_0)}_{\Delta p}(p_m + p_0) \approx \Delta p(2p_0)$$

$$\therefore \frac{\Delta p}{p} = \frac{p_m^2 - p_0^2}{2p_0^2} = \frac{-2m_\nu^2(m_\pi^2 + m_\mu^2)}{4m_\pi^2} \times \frac{1}{2}\frac{4m_\pi^2}{(m_\pi^2 - m_\mu^2)^2}$$

from ⊛⊛

(b)

$$\Rightarrow \boxed{\frac{\Delta p}{p} \approx -\frac{m_\nu^2(m_\pi^2 + m_\mu^2)}{(m_\pi^2 - m_\mu^2)^2}} \approx -\frac{4 m_\nu^2 (\text{MeV})}{10^4}$$

using values of m_π, m_μ

Q.E.D

Find values for **α, β** and **γ** that give three normalized mutually orthogonal wavefunctions with positive lobes directed at 120° with respect to each other in the xy plane.

xxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxx

SOLUTIONS:

A)
The corners of a tetrahedron are in the directions [1 1 1], [-1 -1 1], [1, -1, -1] and [-1 1 -1]: the sets of coefficients (a_x, a_y, a_z) must be proportional to these vectors. For normalization:

$$\int |\psi|^2 \, dV = a_x^2 + a_y^2 + a_z^2 = 1,$$

since the p states are orthonormal. The required sets of coefficients are therefore (1/√3, 1/√3, 1/√3), (-1/√3, -1/√3, 1/√3), (1/√3, -1/√3, -1/√3), (-1/√3, 1/√3, -1/√3).

B)
For the first of these linear combinations we have $\varphi_{111} = bs + c/\sqrt{3}\,(p_x + p_y + p_z)$.

For normalization

$$\int |\varphi_{111}|^2 \, dV = b^2 + c^2 = 1$$

For orthogonality, for example

$$\int \varphi_{111}\, \varphi_{-1-11} \, dV = 0$$

That is

$$\int b^2 s^2 \, dV + \int c^2/3\,(p_x + p_y + p_z)(-p_x - p_y + p_z)\, dV = b^2 - c^2/3 = 0$$

These are satisfied by $b = 1/2$, $c = \sqrt{3}/2$ so that $\varphi_{111} = \frac{1}{2}(s + p_x + p_y + p_z)$. The others can be calculated similarly.

C).
Normalized vectors **(β,γ)** at 120° to each other are **(1,0), (-1/2, √3/2)** and (- **(1/2, -√3/2)**.

Consider unnormalized states,

$\xi_1 = \alpha s + p_x$ and $\xi_2 = \alpha s - 1/2\, p_x + \sqrt{3}/2\, p_y$. ($\alpha$ may be taken the same in both cases because the wavefunctions differ only in orientation).

For orthogonality

$$\int \xi_1\, \xi_2\, dV = \alpha^2 - 1/2 = 0$$

so that $\alpha = 1/\sqrt{2}$.

For normalization

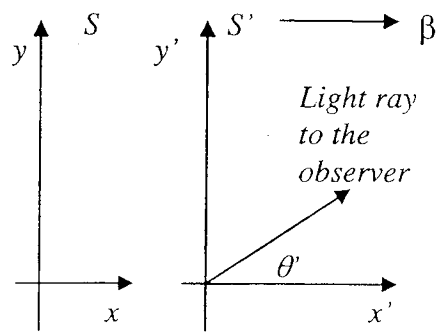
$$\int |\xi_1|^2\, dV = \int |s/\sqrt{2} + p_x|^2\, dV = 3/2,$$

so that for normalization all states must be multiplied by $\sqrt{(3/2)}$.

The required values of (α, β, γ) are $(1/\sqrt{3}, \sqrt{(3/2)}, 0)$, $(1/\sqrt{3}, -1/\sqrt{6}, 1/\sqrt{2})$, and $(1/\sqrt{3}, -1/\sqrt{6}, -1/\sqrt{2})$. The other orthogonalities and normalizations can be produced and checked in a similar way.

Sec 4 #4

Quals Question – Relativity
Mike Tuts, 11/22/06



Question: Consider the usual S and S' frames as shown. A ray of light of frequency ν' is emitted from the origin of the S' frame at an angle θ' towards an observer who is at rest in S.

A. Derive an expression for frequency of the light as observed in S, i.e. find ν in terms of ν', β, θ'.

B. Your freshman students have learned about the Doppler effect so they would not be surprised to observe a change the frequency of the light from a moving source even if they knew nothing about relativity. But based on what you have derived above, under what conditions is there a Doppler effect observed that is purely relativistic effect?

Answer:

A. Consider a plane wave in S' propagating along the direction shown in the plot, i.e. along θ'. Such a wave would be described by

$$\cos 2\pi\left(\frac{x'\cos\theta'+y'\sin\theta'}{\lambda'}-\nu't'\right)$$

which has a velocity $c = \lambda'\nu'$.

Whereas in the S frame it is still a plane wave (since the Lorentz transformation is linear) described in a similar fashion

$$\cos 2\pi\left(\frac{x\cos\theta+y\sin\theta}{\lambda}-\nu t\right)$$

So now apply the Lorentz transformations

$$x'=\frac{x-vt}{\sqrt{1-\beta^2}},\ y'=y,\ t'=\frac{t-(\frac{v}{c^2})x}{\sqrt{1-\beta^2}}$$

To obtain

$$\cos 2\pi\left(\frac{\cos\theta'+\beta}{\lambda'\sqrt{1-\beta^2}}x+\frac{\sin\theta'}{\lambda'}y-\frac{(1+\beta\cos\theta')}{\sqrt{1-\beta^2}}t'\right)$$

So comparing terms we get

$$\nu=\frac{\nu'(1+\beta\cos\theta')}{\sqrt{1-\beta^2}}$$

B. If the source is moving transverse (θ=90^o) to the observer, then classically there would be NO Doppler shift. In the relativistic case, there IS a Doppler shift because of the time dilation. Formally, take the expression above and write the inverse transformation as

$$\nu' = \frac{\nu(1-\beta\cos\theta)}{\sqrt{1-\beta^2}}$$

Which we can solve for ν with θ=90^o

$\nu = \nu'\sqrt{1-\beta^2}$ (which reduces to the classical case of no shift for β small).

[illegible] a crystal [illegible]. Let $\vec{a}_1$, $\vec{a}_2$ and $\vec{a}_3$ define the distance between atoms in directions x_1, x_2 and x_3. The crystal has N_1, N_2 and N_3 atoms in directions x_1, x_2 and x_3 respectively so the position of an atom in the crystal can be written:

$$\vec{r}_j = n_1 \vec{a}_1 + n_2 \vec{a}_2 + n_3 \vec{a}_3$$

Incident on the crystal is a plane wave [illegible]

$$e^{i(\vec{k}\cdot\vec{r} - \omega t)}.$$

[illegible]

Assuming that the incoming wave is scattered elastically by the atoms in the crystal, what is the form of the intensity of the scattered wave at a distant point P?

[illegible]

Note: $\sum_{j=0}^{N-1} e^{j\alpha} = \frac{1 - e^{N\alpha}}{1 - e^{\alpha}}$