Columbia University
Department of Physics
QUALIFYING EXAMINATION
Monday, January 9, 2006
9:00 AM – 11:00 AM

Classical Physics
Section 1. Classical Mechanics

Two hours are permitted for the completion of this section of the examination. Choose 4 problems out of the 5 included in this section. (You will not earn extra credit by doing an additional problem). Apportion your time carefully.

Use separate answer booklet(s) for each question. Clearly mark on the answer booklet(s) which question you are answering (e.g., Section 1 (Classical Mechanics), Question 1; Section 1(Classical Mechanics) Question 3, etc.)

Do **NOT** write your name on your answer booklets. Instead clearly indicate your **Exam Letter Code**

You may refer to the single handwritten note sheet on 8 ½ x 11" paper (double-sided) you have prepared on Classical Physics. The note sheet cannot leave the exam room once the exam has begun. This note sheet must be handed in at the end of today's exam. Please include your Exam Letter Code on your note sheet. No other extraneous papers or books are permitted.

Simple calculators are permitted. However, the use of calculators for storing and/or recovering formulae or constants is NOT permitted.

Questions should be directed to the proctor.

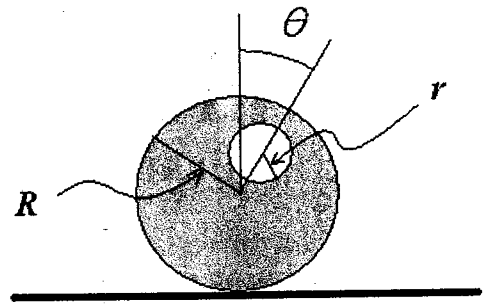
Good luck!!

Problem 1 : Section 1 Classical Mechanics

A cylinder of length L, radius R and mass density ρ rolls on a horizontal surface without slipping. A hole of radius r<R has been drilled through the cylinder parallel to its axis at a distance R/2 from its center. Describe the orientation of the cylinder by specifying the angle θ between the vertical direction and a line connecting the centers of the cylinder and the hole. If initially the cylinder is at rest but θ has a small non-zero value,

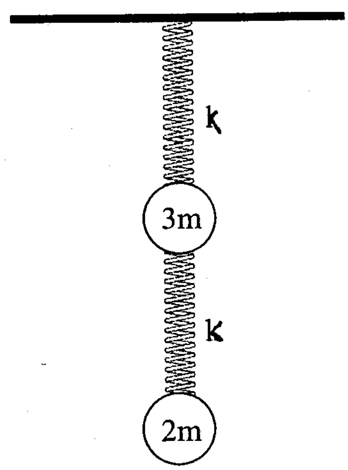
$$\theta\,(t=0) = \theta_0 \ll 1$$

predict the subsequent motion θ (t). Draw a graph of θ (t) indicating the times, if any, where θ =0.



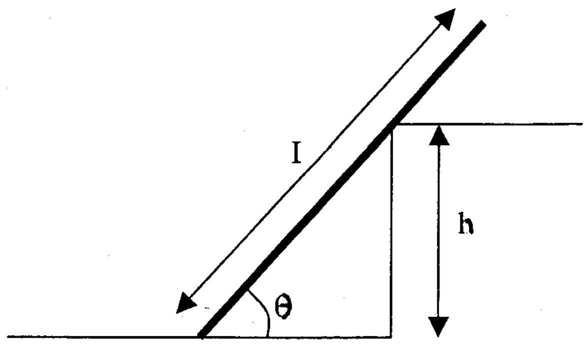
Problem 2 : Section 1 Classical Mechanics

Two massless springs with spring constant k are connected to two masses that hang vertically as shown in the figure. The top one has mass 3m and the bottom one has mass 2m. Find the frequencies of the normal modes of this system for vertical displacements. Describe the motion of each of the normal modes.



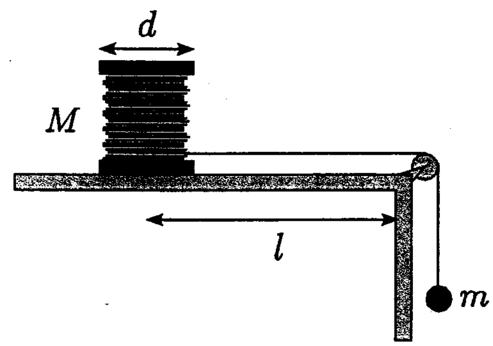
Problem 3 : Section 1 Classical Mechanics

A uniform ladder of weight W and length L is leaning at an angle θ against a structure whose height is h < L. The situation is pictured in the figure below. (Note that the normal force at the corner is perpendicular to the ladder.) There is static friction between the ladder and the ground, but negligible friction between the ladder and the vertical structure. Find the coefficient of friction between the ladder and ground that would be necessary to keep the ladder from moving in terms of L, h, and θ.



Problem 4 : Section 1 Classical Mechanics

A solid spool of mass M and diameter d is released from rest a distance l from the edge of a table. The spool is connected via a massless, inextensible string to a hanging mass m. The spool slides and rotates freely. What is the velocity of the mass m when the spool's center of mass reaches the edge of the table?



Problem 5 : Section 1 Classical Mechanics

Consider the motion of the earth around the sun. Let's approximate the orbit as circular. Suppose the sun very slowly loses its mass, from an original mass of M_1 to a mass of M_2. Suppose the initial radius of the orbit is R_1 and the final radius is R_2.
What is R_2 in terms of the other parameters?

Sec 1 # 1

N. Christ November 26, 2005

Quals Problems

1. Quantum Mechanics:

Consider a hydrogen atom in the 1s state. The magnetic interaction of the spin of the proton $\vec{s}_P$ and that of the electron $\vec{s}_e$ is given by the hyperfine Hamiltonian:

$$H_{\rm HF} = +\frac{8\pi}{3}\frac{g_P g_e}{4m_P m_e c^2}\,\vec{s}_P\cdot\vec{s}_e\,\delta^3(\vec{r}_e) \qquad (1)$$

where $\vec{r}_e$ is the relative coordinate of the electron, g_e and g_P the g-factors for the electron and proton and m_P and m_e their respective masses.

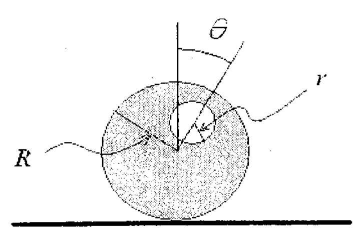
(a) If the hydrogen atom wave function is $\psi(\vec{r}) = e^{-r/a_0}/\sqrt{\pi a_0^3}$ with $a_0 = \hbar^2/(m_e e^2)$, find the splitting between the $F = 0$ and $F = 1$ hyperfine states. (Here $\hbar\vec{F}$ is the total spin of the electron and proton.) [8 points]

(b) If a weak magnetic field $\vec{B}$ is applied, determine the shift in the energy, $\delta E(B)$, of the lowest hyperfine state. [10 points]

(c) Compute the magnetic polarizability, $\alpha_B = -\partial^2\delta E(B)/\partial B^2|_{B=0}$ for this ground state. [2 points]

✗ 2. Mechanics:

A cylinder of length L, radius R and mass density ρ rolls on a horizontal surface without slipping. A hole of radius $r < R$ has been drilled through the cylinder parallel to its axis at a distance $R/2$ from its center. Describe the orientation of the cylinder by specifying the angle θ between the vertical direction and a line connecting the centers of the cylinder and the hole. If initially the cylinder is at rest but θ has a small non-zero value, $\theta(t = 0) = \delta\theta$, describe the subsequent motion. Find the time required for θ to decrease to zero. [20 points]



Sec 1 # 1
Christ
p. 2
(solution)

* 2. Consider rotation about the point of contact, P. Treat the cylinder as a complete cylinder of radius R with mass $M == \rho\pi R^2 L$ and a second of negative mass $-m = -\rho\pi r^2 L$. The first cylinder exerts no torque about P while the second exerts:

$$\tau = -\frac{3R}{2}\rho\pi r^2 Lg\theta \quad [5 \text{ points}] \tag{7}$$

assuming θ to be small.

The moment of inertia about P is that of the cylinder of radius R minus that of r:

$$I = \frac{1}{2}MR^2 + MR^2 - \frac{1}{2}mr^2 - m(3R/2)^2 \quad [5 \text{ points}] \tag{8}$$

where the parallel axis theorem has been used.

Finally we can combine these:

$$I\frac{d^2\theta}{dt^2} = \tau = -\frac{3R}{2}\rho\pi r^2 Lg\theta \quad [5 \text{ points}] \tag{9}$$

which describes simple harmonic motion with period

$$T = \sqrt{2I/3R\rho\pi r^2 Lg} \quad [3 \text{ points}] \tag{10}$$

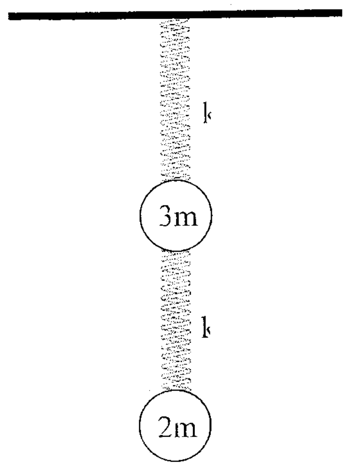
Thus, the cylinder will roll back and forth, executing simple harmonic motion about the equilibrium position $\theta = 0$. It will take $T/4$ time units to first reach $\theta = 0$ [2 points].

SCC 1 # 2

Quals Problem 2 Classical Mechanics

M. Shaevitz
2006
11/28/05

Two massless springs with spring constant k are connected to two masses that hang vertically as shown in the figure. The top one has mass 3m and bottom one has mass 2m. Find the frequencies of the normal modes of this system for vertical displacements. Describe the motion of each of the normal modes.



Solution:

Let $x_1(x_2)$ be the position of the top (bottom) mass with respect to the ceiling.

$$L=\frac{1}{2}3m\dot{x}_1^2+\frac{1}{2}2m\dot{x}_2^2+3mgx_1+2mgx_2-\frac{1}{2}kx_1^2-\frac{1}{2}k\left(x_2-x_1\right)^2$$

Then Lagrange's equations are:

$$3m\ddot{x}_1-3mg+2kx_1-kx_2=0$$

$$2m\ddot{x}_2-2mg+kx_2=0$$

The mg factors can be removed with a change of variables.

Assuming small oscillations with $x_i=A_i\cos\omega t$ gives

$$\begin{pmatrix}2k-3m\omega^2 & -k\\ -k & k-2m\omega^2\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix}=0$$

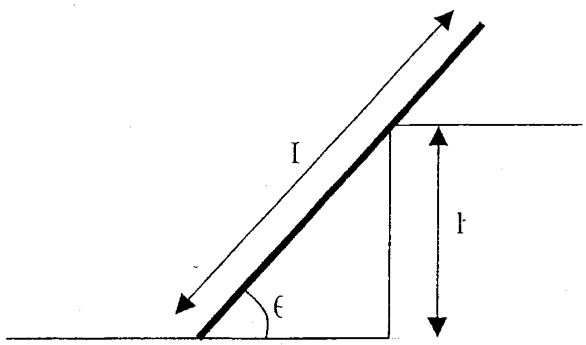
which yields nomal mode frequencies of $\sqrt{k/m}$ and $\sqrt{k/6m}$

For the $\sqrt{k/m}$ frequency the motion has both masses moving in opposite directions with $x_1=-x_2$ and for the $\sqrt{k/6m}$ frequency the motion has both masses moving in same direction with $x_1=\frac{3}{2}x_2$.

Quals Problem 3 Classical Mechanics

M. Shaevitz
2006
11/28/05

A uniform ladder of weight W and length L is leaning at an angle θ against a structure whose height is h < L. The situation is pictured in the figure below. (Note that the normal force at the corner is perpendicular to the ladder.) There is static friction between the ladder and the ground, but negligible friction between the ladder and vertical structure. Find the coefficient of friction between the ladder and ground that would be necessary to keep the ladder from moving in terms of L, h, and θ.



Solution:

Let N_1 be the upward normal force of the ground and N_2 be the normal force from the vertical corner.

Vertical Forces: $N_1 + N_2 \cos\theta - W = 0$

Horizontal Forces: $-N_2 \sin\theta + f = 0$

Torques around ground point: $-W\frac{L}{2}\cos\theta + N_2\frac{h}{\sin\theta} = 0$

Solving these gives:

$$N_2 = \frac{WL\sin\theta\cos\theta}{2h} \qquad N_1 = \frac{W(2h - L\sin\theta\cos^2\theta)}{2h} \qquad f = \frac{WL\sin^2\theta\cos\theta}{2h}$$

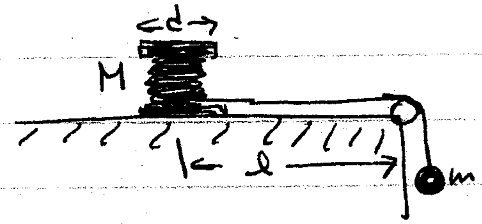
Then $\mu = f / N_1 = \dfrac{L\sin^2\theta\cos\theta}{(2h - L\sin\theta\cos^2\theta)}$

Amber Miller

NOV … 2005 Sec. 1 #4

Mechanics

Problem - A solid spool of mass M and diameter d is released from rest a distance l from the edge of a table. The spool is connected via a massless, inextensible string to a hanging mass m. The spool slides and rotates freely. What is the velocity of the mass m when the spools center of mass reaches the edge of the table?



Solution

$$\tau = I\ddot{\theta} = T\frac{d}{2} \qquad T = M\ddot{x} = mg - m\ddot{x} - m\frac{d}{2}\ddot{\theta}$$

$$\ddot{\theta}\left[\frac{1}{2}M\left(\frac{d}{2}\right)^2\right] = T\frac{d}{2}$$

$$T = M\frac{d}{4}\ddot{\theta}$$

$$\ddot{\theta} = \frac{4T}{Md} = \frac{4}{Md}(M\ddot{x})$$

$$= \frac{4}{d}\ddot{x}$$

$$M\ddot{x} = mg - m\ddot{x} - m\frac{d}{2}\left(\frac{4}{d}\ddot{x}\right)$$

$$= mg - m\ddot{x} - 2m\ddot{x}$$

$$= mg - 3m\ddot{x} \quad \Rightarrow \quad \ddot{x} = \frac{mg}{M+3m}$$

time to get to edge of table $x = \frac{1}{2}at^2 = l$

$$t = \sqrt{\frac{2l}{a}} = \sqrt{\frac{2l}{\ddot{x}}}$$

$$t = \sqrt{\frac{2l(M+3m)}{mg}}$$

velocity of m $v = at = (\ddot{x} + \frac{d}{2}\ddot{\theta})t$

$$= \left[\frac{mg}{M+3m} + \frac{d}{2}\left(\frac{4}{d}\right)\left(\frac{mg}{M+3m}\right)\right]t$$

$$= \frac{mgt}{M+3m}(1+2)$$

$$= \frac{3mgt}{M+3m} = \frac{3mg}{M+3m}\sqrt{\frac{2l(M+3m)}{mg}}$$

$$= \sqrt{\frac{2l(M+3m)}{mg}\,9\left[\frac{mg}{M+3m}\right]^2}$$

$$\boxed{v = \sqrt{\frac{18lmg}{(M+3m)}}}$$

mechanics #5

CORRECTED VERSION

Subject: 2 questions for the Quals committee
From: Lam Hui <lhui@astro.columbia.edu>
Date: Wed, 23 Nov 2005 12:42:20 -0500 (EST)
To: lalla@phys.columbia.edu, lhui@phys.columbia.edu

To the Quals Committee,

Below please find two problems: one mechanics and one general.

Lam

Mechanics -

Problem:

Consider the motion of the earth around the sun. Let's approximate the orbit as circular. Suppose the sun very slowly loses its mass, from an original mass of M_1 to a mass of M_2. Suppose the initial radius of the orbit is R_1 and the eventual radius is R_2. What is R_2 in terms of the other parameters?

Solution:

The angular momentum is an adiabatic invariant. Therefore, $M_1 R_1 v_1 = M_2 R_2 v_2$, with $v_1 = \sqrt{G M_1/R_1}$ and $v_2 = \sqrt{G M_2/R_2}$. Hence, $R_2 = R_1 (M_1/M_2)$ i.e. the orbit expands under mass loss.

Columbia University
Department of Physics
QUALIFYING EXAMINATION
Monday, January 9, 2006
11:10 AM – 1:10 PM

Classical Physics
Section 2. Electricity, Magnetism & Electrodynamics

Two hours are permitted for the completion of this section of the examination. Choose 4 problems out of the 5 included in this section. (You will not earn extra credit by doing an additional problem). Apportion your time carefully.

Use separate answer booklet(s) for each question. Clearly mark on the answer booklet(s) which question you are answering (e.g., Section 2 (Electricity etc.), Question 2; Section 2(Electricity etc.) Question 4, etc.)

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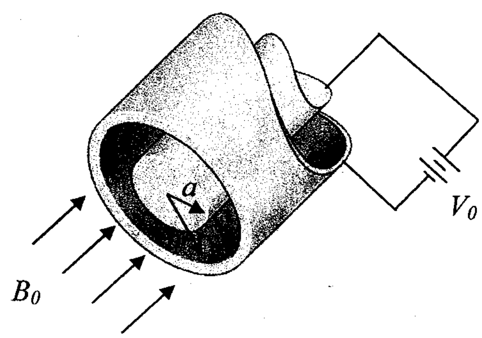
Simple calculators are permitted. However, the use of calculators for storing and/or recovering formulae or constants is NOT permitted.

Questions should be directed to the proctor.

Good luck!!

Problem 1 : Section 2 EM

A very long conducting cylindrical rod of radius a and length L is surrounded by a conducting cylindrical shell whose inner radius is b. There is electric potential V_0 is applied between two conductors (the inner conductor is at higher potential) and a uniform magnetic field B_0 is directed along the axis of the cylinder as shown in the figure below.



(a) Find the total net charge on the inner conductor.

(b) Suppose an electron with charge $-e$ and mass m is orbiting around a circular orbit around the inner conductor at a distance R away from its cylindrical axis and well away from the edge of cylinders. Find the velocity v of the electron in this circular orbit.

Problem 2 : Section 2 EM

An oscillating electric dipole moment $\vec{p}(t) = p_0 \cos(wt)\,\hat{z}$ generates radiating electric and magnetic fields. Far away from the dipole, the scalar, $V(\vec{x},t)$, and vector potentials $\vec{A}(\vec{x},t)$, due to this dipole are written as

$$V = -\frac{p_0 w}{4\pi\varepsilon_0 c}\left(\frac{\cos\theta}{r}\right)\sin[w(t-r/c)] \quad \text{and} \quad \vec{A} = -\frac{\mu_0 p_0 w}{4\pi r}\sin[w(t-r/c)]\,\hat{z}$$

in SI unit where $c^2 = 1/(\mu_0\varepsilon_0)$

(a) Show that the total find power of radiation emitted from this dipole is given by $P = \dfrac{\mu_0 p_0^2 w^4}{12\pi c}$ in SI unit (or $P = \dfrac{p_0^2 w^4}{3c^3}$ in cgs unit).

(Hint: Work in spherical coordinates. This integral might be useful $\int_0^\pi \sin^3\theta\, d\theta = \frac{4}{3}$).

(b) Consider a classical charged simple harmonic oscillator with mass m and charge q is oscillating with angular frequency w. Let A_0 is the oscillation amplitude at t =0. Find the time , $T_{1/2}$, when the amplitude of the oscillator reduces in half.

Problem 3 : Section 2 EM

Maxwell's equations yield the following wave equations for a linear, isotropic medium with conductivity σ:

$$\nabla^2 \vec{E} - \epsilon\mu \frac{\partial^2 \vec{E}}{\partial t^2} - \sigma\mu \frac{\partial \vec{E}}{\partial t} = \frac{1}{\epsilon}\vec{\nabla}\rho_f \qquad (1)$$

$$\nabla^2 \vec{H} - \epsilon\mu \frac{\partial^2 \vec{H}}{\partial t^2} - \sigma\mu \frac{\partial \vec{H}}{\partial t} = 0 \qquad (2)$$

with

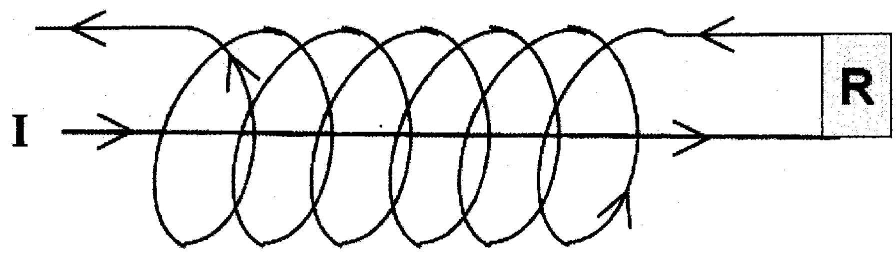
$$\mu \frac{\partial \vec{H}}{\partial t} + \vec{\nabla} \times \vec{E} = 0 \qquad (3)$$

Consider a plane polarized electromagnetic wave in vacuum, propagating in the positive z direction. It strikes a semi-infinite conducting slab, whose boundary is at $z = 0$. Determine the ratio of the amplitude for the reflected wave to that of the incident wave for the case where the conducting slab is a good conductor ($\sigma >> \omega\epsilon$).

Problem 4 : Section 2 EM

Steady current I flows in the circuit below. The solenoid is long with length L >> radius a, and number of turns n=N/L >> 1/a. The resistance R is given but the resistivity of the wire elsewhere can be neglected. The straight wire inside the solenoid is coaxial with the solenoid.

Find the net flux of electromagnetic energy through the cross section area π a^2 , of the solenoid (far from its edges)

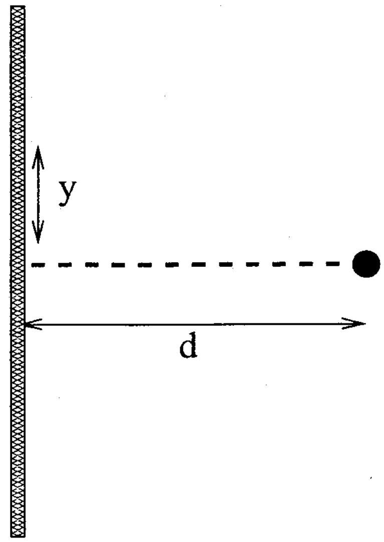


Problem 5 : Section 2 EM

A very long wire of radius a is suspended a distance d above an infinite conducting plane. In the case that $d >> a$, find approximate expressions for

a. The capacitance per unit length of the wire, conducting plane system.

b. The surface charge density on the conducting plane as a function of y, the distance along the plane lateral to the wire.

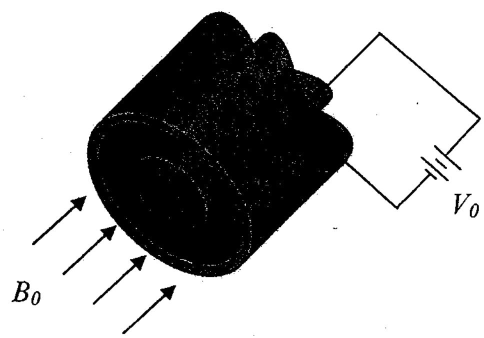


Scc2 #1

Philip Kim 2006 Qual

E&M I:

A very long conducting cylindrical rod of radius a and length L is surrounded by a conducting cylindrical shell whose inner radius is b. There is electric potential V_0 is applied between two conductors (the inner conductor is at higher potential) and a uniform magnetic field B_0 is directed along the axis of the cylinder as shown in the figure below.



(a) Find the total net charge on the inner conductor.

(b) Suppose an electron with charge $-e$ and mass m is orbiting around a circular orbit around the inner conductor at a distance R away from its cylindrical axis and well away from the edge of cylinders. Find the velocity v of the electron in this circular orbit.

Sec 2 #1
Kim-solution

Kim

E&M I Sol

(a) The potential between the conductor

$$V(r) = -\frac{V_0 \ln(\frac{r}{a})}{\ln(\frac{b}{a})} \Rightarrow \text{Electric field} \quad \vec{E} = \frac{V_0}{\ln(\frac{b}{a})}\frac{1}{r}\hat{r}$$

At the surface of the inner conductor,

$$\sigma = \hat{r}\cdot\vec{E}\Big|_{r=a} = \frac{V_0}{\ln(\frac{b}{a})}\frac{1}{a}.$$

Thus the total net charge on the inner conductor

$$Q = L\cdot 2\pi a\,\sigma = \frac{2\pi V_0 L}{\ln(b/a)}$$

(b) For a circular motion of radius R, considering electrostatic & Lorentz force

$$\frac{mv^2}{R} = e\frac{V_0}{\ln(b/a)}\frac{1}{R} + evB_0$$

$$\Rightarrow \quad v^2 - \left(\frac{eB_0}{m}R\right)v - \frac{eV_0}{m\ln(b/a)} = 0$$

or

$$v = \omega_L R \pm \sqrt{(\omega_L R)^2 + \frac{eV_0}{m\ln(b/a)}}$$

where $\omega_L = \frac{eB_0}{2m}$

DEC 5 2005 Sec 2 #2

Philip Kim 2006 Qual

E&M II:

An oscillating dipole moment $\vec{p}(t) = p_0 \cos(wt)\,\hat{z}$ generates radiating electric and magnetic field. At far away from the dipole, the vector potential due to this dipole is written as

$\vec{A} = -\frac{\mu_0 p_0 w}{4\pi r}\sin[w(t - r/c)]\hat{z}$ in SI unit (or $\vec{A} = -\frac{p_0 w}{cr}\sin[w(t - r/c)]\hat{z}$ in cgs unit).

(a) Show that the total find power of radiation emitted from this dipole is given by

$P = \frac{\mu_0 p_0^2 w^4}{12\pi c}$ in SI unit (or $P = \frac{p_0^2 w^4}{3c^3}$ in cgs unit).

(This integral might be useful $\int_0^\pi \sin^3\theta\, d\theta = \frac{4}{3}$).

(b) Consider a classical charged simple harmonic oscillator with mass m and charge q is oscillating with angular frequency w. Let A_0 is the oscillation amplitude at t =0. Find the time where the amplitude of the oscillator reduces in half.

Sec 2 #2

(2)

(b) Let $X(t) = A_0 \cos\omega t$ is the position of the charge

Then the charge density is described by

$$\rho(x) = q\,\delta(X(t)) = q\,\delta(A_0\cos\omega t)$$
$$= +q\,\delta(0) - q\delta(0) + q\,\delta(A_0\cos\omega t)$$
$$= +q\,\delta(0) + P_0\cos\omega t$$

where $P_0 = q A_0$

Since the static charge at $x=0$ does not radiate, thus energy of the SHO is reduced by dipole radiation.

Energy of the SHO

$$E = \frac{m}{2}\omega^2 A_0^2$$

From the result of (a),

$$\frac{dE}{dt} = -P = -\frac{\mu_0\omega^4}{12\pi c}(qA_0)^2 \qquad \text{Here } \frac{dE}{dt} = m\omega^2 A_0\frac{dA_0}{dt}.$$

or

$$m\omega^2 A_0\frac{dA_0}{dt} = -\frac{\mu_0\omega^4}{12\pi c}q^2 A_0^2$$

$$\Rightarrow \qquad \frac{dA_0}{dt} = -\frac{A_0}{\tau} \qquad \text{where } \boxed{\tau = \frac{12\pi c\, m}{\mu_0\omega^2 q^2}}$$

or

$$A_0(t) = A_0(0)\,e^{-t/\tau}$$

The amplitude reduces in half

when $t = \tau \ln 2$.

Kim

E&M II sol

(1)

(a)

$$\vec{A} = -\frac{\mu_0 P_0 \omega}{4\pi r}\sin[\omega(t-\tfrac{r}{c})]\,\hat{z} \qquad \hat{z} = \cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}$$

$$\vec{B} = \vec{\nabla}\times\vec{A} = \left[\frac{1}{r}\frac{\partial}{\partial r}(rA_\theta) - \frac{1}{r}\frac{\partial A_r}{\partial\theta}\right]\hat{\phi}$$

$$= -\frac{\mu_0 P_0 \omega}{4\pi}\left\{\frac{\omega}{c}\frac{1}{r}\sin\theta\cos[\omega(t-\tfrac{r}{c})] + \frac{\sin\theta}{r^2}\sin[\omega(t-\tfrac{r}{c})]\right\}\hat{\phi}$$

$\sim \frac{1}{r^2}$ not propagating ignore !

$$\simeq -\frac{\mu_0 P_0 \omega}{4\pi}\frac{\omega}{cr}\sin\theta\cos[\omega(t-\tfrac{r}{c})]\hat{\phi}$$

Since $\vec{E}$ field is orthogonal to $\vec{B}$ & $\hat{r}$, and

$$|\vec{B}|/|\vec{E}| = \frac{1}{c},$$

Poynting vector $\vec{S} = \frac{1}{\mu_0}\vec{E}\times\vec{B} = \frac{1}{\mu_0 c}|\vec{B}|^2\,\hat{r}$

$$\langle\vec{S}\rangle = \frac{1}{\mu_0 c}\left(\frac{\mu_0 P_0 \omega^2}{4\pi c r}\right)^2\sin^2\theta\cdot\frac{1}{2}\hat{r} = \frac{\mu_0 P_0^2 \omega^4}{32\pi^2 c}\frac{\sin^2\theta}{r^2}\hat{r}$$

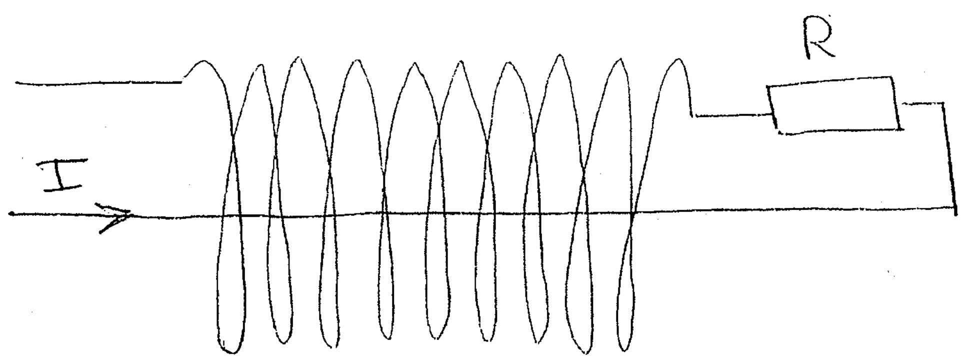
time average

$$P = \int_0^{2\pi}\int_0^{\pi}\langle\vec{S}\rangle\cdot\hat{r}\; r^2\sin\theta\, d\theta\, d\phi$$

$$= \frac{\mu_0 P_0^2\omega^4}{32\pi^2 c}\cdot 2\pi\cdot\underbrace{\int_0^{\pi}\sin^3\theta\, d\theta}_{=4/3} = \frac{\mu_0 P_0^2 \omega^4}{12\pi c},$$

Sec 2 #4

Belobordov
11/28/05 E&M



Problem 1.

Steady current I flows in the cicruit shown in the figure. The solenoid is long (length $L \gg$ radius a) and has number of turns $n = N/L \gg a^{-1}$. The resistance R is given; neglect resistivity of the wire everywhere else in the circuit. The straight wire inside the solenoid is coaxial with the solenoid. Find the net flux of electromagnetic energy through the πa^2 cross section of the solenoid (far from its edges).

Solution: Poynting flux inside the solenoid is

$$\mathbf{S} = \frac{c}{4\pi}\mathbf{E}\times\mathbf{B}.$$

We'll use cylindrical coordinates r, ϕ, z with the z-axis along the axis of the solenoid. First find electric field $\mathbf{E}$. It is created bceause there is potential drop IR between the solenoid and the wire on its axis. By symmetry, $E_\phi = E_z = 0$, and the non-zero component E_r depends on r only. E_r may be found from $\nabla\cdot\mathbf{E} = 0$ between the wire and the solenoid, which gives

$$\frac{1}{r}\frac{d}{dr}(rE_r) = 0, \qquad E_r = \frac{C}{r}.$$

C is found from the known potential drop. Denote the radius of the wire by b, then

$$IR = \int_b^a E_r\, dr = C\ln\frac{a}{b}, \qquad C = \frac{IR}{\ln(a/b)}.$$

The Poynting flux is then given by,

$$\mathbf{S} = \frac{c}{4\pi}E_r\mathbf{e}_r \times (B_\phi\mathbf{e}_\phi + B_z\mathbf{e}_z) = \frac{c}{4\pi}E_r\,(B_\phi\mathbf{e}_z - B_z\mathbf{e}_\phi),$$

where $\mathbf{e}_r$, $\mathbf{e}_\phi$, and $\mathbf{e}_z$ are unit vectors tangent to the coordinates lines and we have used $\mathbf{e}_r\times\mathbf{e}_\phi = \mathbf{e}_z$ and $\mathbf{e}_r\times\mathbf{e}_z = -\mathbf{e}_\phi$. The net flux of electromagnetic energy through the solenoid is

$$\mathbf{F} = \int_b^a dr\int_0^{2\pi} d\phi\,\mathbf{S} = \int_b^a \frac{c}{4\pi}E_r B_\phi \mathbf{e}_z 2\pi r\, dr \qquad (1)$$

(the second term with $B_z\mathbf{e}_\phi$ vanishes after integration by symmetry). It remains to find $B_\phi(r)$ and calculate the integral (1).

The solenoid itself creates a uniform B_z and does not contribute to B_ϕ. The axial wire creates B_ϕ which is found by integrating Maxwell equation $\nabla\times\mathbf{B} = (4\pi/c)\mathbf{j}$ over the cross section of the wire and then applying the Stokes' theorem,

$$2\pi r B_\phi = \frac{4\pi}{c}I, \qquad B_\phi = \frac{2I}{cr}.$$

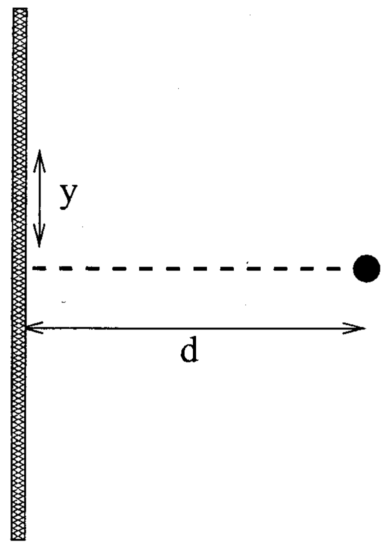
Substituting the known $E_r(r)$ and $B_\phi(r)$ into equation (1) and performing the integration, one finds

$$F = I^2R.$$

A very long wire of radius a is suspended a distance d above an infinite conducting plane. In the case that $d >> a$, find approximate expressions for

a The capacitance per unit length of the wire, conducting plane system.

b The surface charge density on the conducting plane as a function of y, the distance along the plane lateral to the wire.



sec 2 #5

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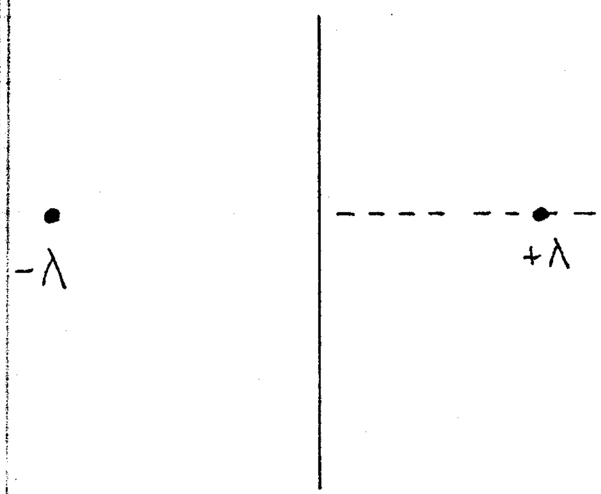
2006 Qualifying Exam

Classical Physics, E&M
Problem 5 Solutions

a) Because $a \ll d$, we can treat the wire as if it is a carrier of charge of linear density λ.

Use the method of images to account for the induced charge on the surface of the conducting sheet, so imagine linear charge density $-\lambda$ a distance d past the conducting sheet.



Then, if we choose the electrostatic potential, φ, to be zero on the sheet, along the line passing through the charges,

$$\Delta \varphi = -\int_0^{d-a} dx \; E(x)$$

$$E(x) = \frac{-\lambda}{2\pi \varepsilon_0}\left(\frac{1}{-x+d} + \frac{1}{x+d}\right)$$

$$\text{So } \Delta\varphi = \frac{\lambda}{2\pi\varepsilon_0}\left(\ln(x+d) - \ln(d-x)\right)\Big|_0^{d-a}$$

$$= \frac{\lambda}{2\pi\varepsilon_0}\left(\ln\left(\frac{2d-a}{d}\right) + \ln\left(\frac{d}{a}\right)\right) = \frac{\lambda}{2\pi\varepsilon_0}\ln\left(\frac{2d-a}{a}\right)$$

The capacitance per unit length is the charge per unit length / $|\Delta\varphi|$

$$\frac{C}{L} = \lambda / |\Delta\varphi| = \frac{2\pi\varepsilon_0}{\ln\left(\frac{2d-a}{a}\right)}$$

b) The magnitude of the electric field from the wire at $+d$ at the surace of the plane is

$$|E_+(y)| = \frac{\lambda}{2\pi\varepsilon_0}\frac{1}{\sqrt{d^2+y^2}}$$

The component $\perp$ to the plane is the above multiplied by $d/\sqrt{d^2+y^2}$.

The components $\parallel$ to the plane from the wire and its image cancel of course & the $\perp$ component is doubled:

$$|E| = \frac{\lambda}{\pi\varepsilon_0}\frac{d}{d^2+y^2} \rightarrow E(y) = -\frac{\lambda}{\pi\varepsilon_0}\frac{d}{d^2+y^2}\hat{x}$$

Then, the charge density is $\sigma = E\varepsilon_0$

So $\sigma(y) = \dfrac{-\lambda d}{\pi(d^2+y^2)}$