Columbia University
Department of Physics
QUALIFYING EXAMINATION
January 14, 2004
9:00 AM – 11:00 AM

Modern Physics
Section 3. Quantum Mechanics

Two hours are permitted for the completion of this section of the examination. Choose 4 problems out of the 6 included in this section. (You will not earn extra credit by doing an additional problem). Apportion your time carefully.

Use separate answer booklet(s) for each question. Clearly mark on the answer booklet(s) which question you are answering (e.g., Section 3 (QM), Question 1; Section 3(QM) Question 5, etc.)

Do **NOT** write your name on your answer booklets. Instead clearly indicate your **Exam Letter Code**

You may refer to the single note sheet on 8 ½ x 11" paper (double-sided) you have prepared on Modern Physics. The note sheet cannot leave the exam room once the exam has begun. This note sheet must be handed in at the end of today's exam. Please include your Exam Letter Code on your note sheet. No other extraneous papers or books are permitted.

Simple calculators are permitted. However, the use of calculators for storing and/or recovering formulae or constants is NOT permitted.

Questions should be directed to the proctor.

Good luck!!

Problem 1

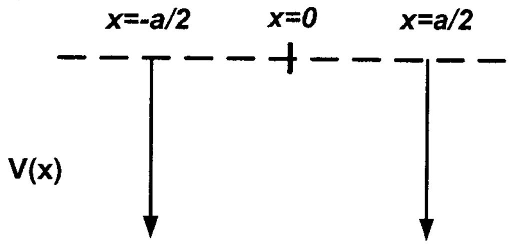
Consider a particle of mass m moving in one dimension over the range $-\infty < x < \infty$ and subject to the attractive two delta-function potential (see sketch),

$V(x) = -V_0\, a\left[\delta(x - a/2) + \delta(x + a/2)\right]$.

Here a is a length and V_0 is a positive constant.

a) For a sufficiently large V_0 this potential has two bound states. Sketch their wave functions.

b) For a less than some critical distance a_c, the potential has only one bound state. Find a_c in terms of m and V_0.



Problem 2

A one-dimensional harmonic oscillator with mass m and spring constant k_0 is subject to a time-independent squeezing perturbation, $V_1(x) = \frac{1}{2} k_1 x^2$, $|k_1 / k_0| << 1$.

a) Using perturbation theory, calculate the ground state energy shift to second order in k_1/k_0.

b) Compare to the exact result for the change in ground state energy in the presence of the perturbation.

c) Compute to first order the perturbed ground state wave function in terms of the unperturbed wave functions $\psi_n^{(0)}(x) = \langle x | n \rangle$.

d) What is the exact ground state wave function, $\psi_0^{\text{exact}}(x)$, in the presence of the perturbation (do not worry about the normalization).

Problem 3

The Hamiltonian for the two-dimensional harmonic oscillator is

$$H = \frac{1}{2m}(p_x^2 + p_y^2) + \frac{1}{2}m\omega^2(x^2 + y^2).$$

As for the one-dimensional harmonic oscillator, one can introduce creation and annihilation operators according to

$$a_x = \sqrt{\frac{m\omega}{2\hbar}}x + i\frac{p_x}{\sqrt{2m\omega\hbar}}$$

$$a_y = \sqrt{\frac{m\omega}{2\hbar}}y + i\frac{p_y}{\sqrt{2m\omega\hbar}}$$

so that $H = \hbar\omega\left(a_x^\dagger a_x + a_y^\dagger a_y + 1\right)$ with $\left[a_x, a_x^\dagger\right] = \left[a_y, a_y^\dagger\right] = 1$. The angular momentum operator is $L = xp_y - yp_x$.

a) Evaluate L in terms of the creation and annihilation operators.

b) Evaluate $\left[L, a_x\right]$ and $\left[L, a_y\right]$.

c) Give a physical argument that requires $[L, H] = 0$. Use the result from part b to check that $[L, H] = 0$.

Problem 4

Consider a particle of mass m moving non-relativistically along the circumference of a horizontal circle of radius R.

a) What are the energy eigenstates and eigenvalues of this system?

b) A uniform magnetic field B is introduced perpendicular to the plane of the circle. If the particle has charge q, find the new energy eigenvalues and eigenstates.

c) Next, consider a system of two neutral particles of mass m_1 and m_2 that move on the circumference of the circle. If an infinite short-range repulsive force prevents them from passing each other, find the resulting energy eigenstates and eigenvalues.

d) Repeat your solution to part c) for the case that the repulsive force is replaced by reflectionless scattering satisfying the boundary condition, $\lim_{\theta_1 \to \theta_2^-} \psi(\theta_1, \theta_2) = e^{i\delta} \lim_{\theta_1 \to \theta_2^+} \psi(\theta_1, \theta_2)$.

Problem 5

An electron in a hydrogen atom is in a state described by the wave function

$$\Psi(\vec{r}) = A\left[\psi_{100}(\vec{r}) + 2\psi_{210}(\vec{r}) + 2\psi_{211}(\vec{r}) - \psi_{21-1}(\vec{r})\right]$$

Here $\psi_{nlm}(\vec{r})$ is a normalized wave function of hydrogen atom with the principle quantum number n, angular quantum number l and magnetic quantum number m, where the explicit functional forms are:

$$\psi_{100}(\vec{r}) = 2a_0^{-3/2}(4\pi)^{-1/2}e^{-r/a_0}$$
$$\psi_{210}(\vec{r}) = (2a_0)^{-3/2}(4\pi)^{-1/2}(r/a_0)e^{-r/2a_0}\cos\theta$$
$$\psi_{21\pm1}(\vec{r}) = \mp(2a_0)^{-3/2}(8\pi)^{-1/2}(r/a_0)e^{-r/2a_0}\sin\theta\, e^{\pm i\varphi}$$

The Bohr radius is $a_0 = \dfrac{\hbar^2}{m_e e^2}$ where m_e is the mass of electron. A is a normalization constant.

a) Neglecting the spin orbit interaction (for now), find the expectation values of the energy, $\boldsymbol{L}^2$ and L_z, where $\boldsymbol{L}$ and L_z are the orbital angular momentum of the hydrogen atom and its z-component, respectively.
b) Now the hydrogen atom described above is placed in a weak gravitational force field, $\vec{F} = -m_e g\,\hat{z}$, where g is the gravitational acceleration constant. Compute the change in the expectation value of the energy to first order in $m_e g$. Assume that the ion core remains fixed in space.
c) We now consider a spin-orbit coupling of the electron in this problem. How many different values of energy level can be measured in the above wave function? (Note that we do not have $\psi_{200}(\vec{r})$ component in the above wave function.)

Problem 6

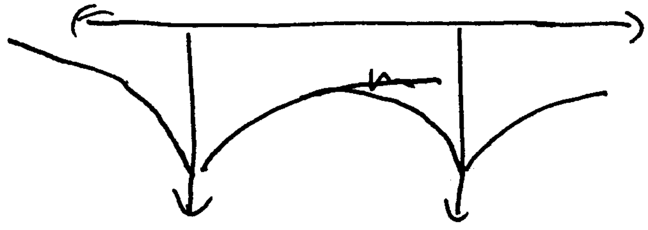
Consider a rigid rotor with moment of inertia, I, and permanent dipole moment, p, located in an external, position independent electric field E. Calculate the ground state energy and wave functions for the following cases:

a) $E=0$
b) $pE << \hbar^2/I$
c) Let the rotor be constrained to the x-y plane. Solve for the ground-state energy and wave function assuming $E = 0$.
d) Let the rotor perform small oscillations ($\phi << 1$) about the x-axis and let $\vec{E} = E_x\hat{x}$. Find the ground-state energy and wave function.

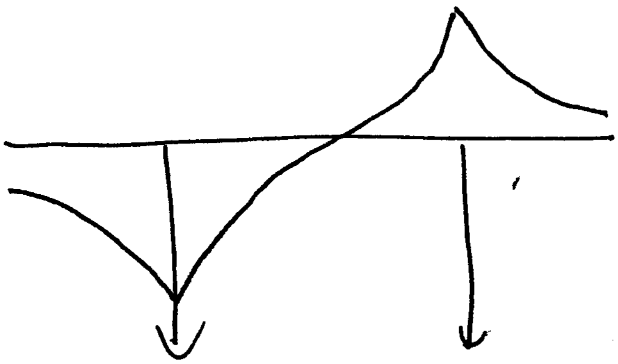
Section 3

Solution Question #1

(i)



Lowest state (even parity)



higher lying bound state (odd parity)

(ii) We seek the odd parity solution of

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V_0 a\left[\delta(x-a/2) + \delta(x+a/2)\right]\psi = E\psi$$

with $E = 0$

$$E = 0 \Rightarrow \psi(x) = A + Bx$$

Soln - 2

Parity $\Rightarrow$

$x > a/2$: $\quad \psi(x) = A$

$-a/2 < x < a/2 \quad \psi(x) = 2A\, x/a$

$x < -a/2 \quad \psi(x) = -A$

Boundary condition at $\delta(x-a/2)$:

$$-\frac{\hbar^2}{2m}\left(\left.\frac{d\psi}{dx}\right|_{x=a/2^+} - \left.\frac{d\psi}{dx}\right|_{x=a/2^-}\right) + V_0 a\,\psi(a/2) = 0$$

$$\Rightarrow \quad \frac{\hbar^2}{ma} + V_0 a = 0 \quad \Rightarrow \quad a = \left(-\frac{\hbar^2}{mV_0}\right)^{1/2}$$

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Quantum (Quals. 04) Section 3, Question # ~~3~~ 2 Gyulassy 1/5

A one dimensional harmonic oscillator with mass m and string constant k_0 is subject to a time independent squeezing perturbation $V_1(x) = \frac{1}{2} k_1 x^2$, $|k_1/k_0| \ll 1$

a) Compute the ground state energy shift through second order in k_1/k_0

b) Compare to exact result for ground state

c) Compute the first order perturbed ground state wavefunction $\psi_0^{(1)}(x)$ in terms of the unperturbed $\psi_n^{(0)}(x) = \langle x | n \rangle$.

d) What is the exact ground state $\psi_0^{exact}(x)$?
(Do not worry about normalization)

A harmonic oscillator $H = p^2/2m + \frac{k_0 x^2}{2}$ is subject to a squeezing perturbation $V_1 = \delta k \, \frac{x^2}{2}$ where $\delta k \equiv k_1$ (notation)

Compute the Ground state energy shift through second order in δk, Compare to exact result

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$$H|n\rangle = E_n^0 |n\rangle \qquad E_n^0 = \hbar\omega_0 (n + 1/2)$$

$\omega_0 = \sqrt{k/m}$ is unpertured ang. freq

$$H = \hbar\omega_0 (a^\dagger a + 1/2)$$

where
$$\hat{a} = \alpha \hat{x} + i \hat{p} \beta$$
$$\hat{a}^\dagger = \alpha \hat{x} - i \hat{p} \beta$$

$$a^\dagger a = \alpha^2 \hat{x}^2 + \beta^2 \hat{p}^2 - i\alpha\beta \underbrace{[\hat{p}, \hat{x}]}_{-i\hbar}$$

$$\hbar\omega_0 \alpha^2 = k_0/2 \qquad \alpha\beta\hbar = 1/2$$
$$\hbar\omega_0 \beta^2 = 1/2m$$

$$\alpha^2 = \frac{k_0}{2\hbar\omega_0} = \frac{\sqrt{k_0 m}}{2\hbar} \qquad \beta^2 = \frac{1}{2\hbar\omega_0 m} = \frac{1}{2\hbar\sqrt{k_0 m}}$$

$$\alpha^2\beta^2 = \frac{k_0}{4m}\frac{1}{\hbar^2\omega_0^2} = \frac{1}{4\hbar^2} \checkmark$$

$$\hat{x} = (a + a^\dagger)/2\alpha \qquad \hat{p} = (\hat{a} - \hat{a}^\dagger)/(2i\beta)$$

$$V_1 = \frac{\delta k}{2} \hat{x}^2 = \frac{1}{2}\frac{\delta k}{4\alpha^2}\left(a^2 + a^{\dagger 2} + a a^\dagger + a^\dagger a\right)$$

$$\frac{\delta k}{4\alpha^2} = \frac{\delta k}{k_0}\frac{\hbar\omega_0}{2} = \left(\frac{k_1}{k_0}\right)\cdot\frac{\hbar\omega_0}{2}$$ 3/5

Perturbation theory to second order

$$E_n = E_n^0 + \langle n|V_1|n\rangle + \sum_{n'\neq n}\frac{\langle n|V_1|n'\rangle\langle n'|V_1|n\rangle}{E_n^0 - E_{n'}^0}$$

Ground state to n

$$\langle n/V_1/0\rangle = \left(\frac{1}{2}\right)\frac{\delta k}{4\alpha^2}\langle n/a^2 + a^{+2} + aa^+ + a^+a/0\rangle$$

$$= \frac{\delta k}{2k_0}\left(\frac{1}{2}\hbar\omega_0\right)\langle n/a^{+2} + aa^+/0\rangle$$

where we used $\boxed{a|0\rangle = 0}$

$$\langle n|aa^+|0\rangle = \langle n|a^+a + 1|0\rangle$$

$$= \delta_{n0}\times 1$$

$$\langle n|a^{+2}|0\rangle = \langle 0|a^2|n\rangle = \sqrt{n}\langle 0|a|n-1\rangle$$

$$= \sqrt{n(n-1)}\underbrace{\langle 0|n-2\rangle}_{\delta_{n2}}$$

$$= \sqrt{2}\,\delta_{n2}$$

$$\langle n/V_1/0\rangle = \left(\frac{\hbar\omega_0}{2}\right)\left(\frac{\delta k}{2k_0}\right)\left(\delta_{n0} + \sqrt{2}\,\delta_{n2}\right)$$

$$E_0 = \frac{1}{2}\hbar\omega_0\left(1 + \frac{1}{2}\frac{\delta k}{k_0}\right) + \underbrace{\left(\frac{\hbar\omega_0}{2}\right)^2\left(\frac{\delta k}{k_0}\right)^2\frac{\sqrt{2}^2}{\frac{1}{2}\hbar\omega_0 - \frac{5}{2}\hbar\omega_0}}_{\frac{1}{2}\hbar\omega_0\left(\frac{\delta k}{2k_0}\right)^2\left(-\frac{1}{2}\right)}$$

$$E_0 = \frac{1}{2}\hbar\omega\left(1 + \frac{1}{2}\frac{\delta k}{k_0} - \frac{1}{8}\left(\frac{\delta k}{k_0}\right)^2 + \cdots\right)$$

b) Compare to exact result

$$E_0^{ex} = \frac{1}{2}\hbar\omega \qquad \text{when} \qquad \omega = \sqrt{\frac{k_0+\delta k}{m}}$$

$$\omega = \omega_0\sqrt{1+\frac{\delta k}{k_0}} = \omega_0\left(1 + \frac{1}{2}\frac{\delta k}{k_0} - \frac{1}{8}\frac{\delta k^2}{k_0^2} + \cdots\right)$$

We recover up to second order the exact result

c.) The perturbed ground state wavefunction

$$|\Psi_0'\rangle \cong |\Psi_0\rangle + \sum_{n>0}\frac{\langle n|V_1|0\rangle}{E_0^0 - E_n^0}|\Psi_n\rangle$$

$$= |0\rangle + \frac{\sqrt{2}\left(\frac{\hbar\omega_0}{2}\right)(\delta k/2k_0)}{\frac{1}{2}\hbar\omega_0 - \frac{5}{2}\hbar\omega_0}|2\rangle$$

enough for full credit

$$= |0\rangle - \sqrt{2}\left(\frac{\delta k}{8k_0}\right)|2\rangle$$

$$\psi_0^{(1)}(x) = e^{-\alpha^2x^2}\left(A_0 - \sqrt{2}\left(\frac{\delta k}{8k}\right)A_2(8\alpha^2x^2-2)\right)$$

$$\frac{A_1}{A_0} = \frac{1}{\sqrt{2}} \qquad \frac{A_2}{A_0} = \sqrt{\frac{1}{2^2\cdot 2!}} = 1/2^{3/2} \qquad A_0 = \frac{\sqrt{\alpha}}{\pi^{1/4}}$$

$$= A_0 e^{-\alpha^2x^2}\left(1 + \frac{1}{8}\frac{\delta k}{k} - \frac{1}{2}\left(\frac{\delta k}{k}\right)\alpha^2x^2\right)$$

Exact $\langle x|\psi_0^{ex}\rangle = \left(\frac{\alpha_{ex}^2}{\pi}\right)^{1/4} e^{-\alpha_{ex}^2 x^2}$.

$$\alpha_{ex}^2 = \frac{\sqrt{(k+\delta k)\, m}}{2\hbar} = \alpha_0^2 \underbrace{\sqrt{1+\delta k/k}}_{1+\frac{1}{2}\frac{\delta k}{k}}$$

$$(\alpha_{ex}^2)^{1/4} \cong (\alpha_0^2)^{1/4}\left(1+\frac{1}{8}\frac{\delta k}{k}\right)$$

not require

d) $\psi_0^{ex}(x) = A\, e^{-\alpha^2 x^2}$

$$\left(-\frac{\hbar^2}{2m}\partial_x^2 + \frac{1}{2}(k_0+k_1)\,x^2\right)\psi_0^{ex} = \frac{1}{2}\hbar\sqrt{\frac{k_0+k_1}{m}}\,\psi_0$$

$$\partial_x \psi = -2\alpha^2 x\,\psi$$

$$\partial_x^2\psi = (+4\alpha^4 x^2 - 2\alpha^2)\,\psi$$

(1) $-\frac{\hbar^2}{2m}4\alpha^4 + \frac{1}{2}(k_0+k_1) = 0$

$$\alpha^2 = \sqrt{\frac{(k_0+k_1)\,m}{4\hbar^2}} = \frac{\sqrt{k_0 m}}{2\hbar}\sqrt{1+k_1/k_0}$$

(2) <u>check</u> $2\alpha^2\frac{\hbar^2}{2m} = \frac{1}{2}\hbar\sqrt{\frac{k_0+k_1}{m}}$

$$\alpha^2 = \frac{1}{2\hbar}\sqrt{(k_0+k_1)m} \qquad \text{ok}$$

not needed

(Norm $\int_{-\infty}^{\infty} A^2 e^{-2\alpha^2 x^2}dx = 1$; $A^2\frac{\sqrt{\pi}}{\sqrt{2\alpha^2}} = 1$)

section 3, Question 2 (3)

A. Mueller

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(1) Quantum (Angular Momentum)

The Hamiltonian for the two-dimensional harmonic oscillator is

$$H = \frac{1}{2m}(p_x^2 + p_y^2) + \frac{1}{2} m\omega^2 (x^2+y^2).$$

As for the one-dimensional oscillator one can introduce creation and annihilation operators according to

$$a_x = \sqrt{\frac{m\omega}{2\hbar}}\, x + i \frac{p_x}{\sqrt{2m\omega\hbar}}$$

$$a_y = \sqrt{\frac{m\omega}{2\hbar}}\, y + i \frac{p_y}{\sqrt{2m\omega\hbar}}$$

so that $H = \hbar\omega(a_x^\dagger a_x + a_y^\dagger a_y + 1)$ with $[a_x, a_x^\dagger] = [a_y, a_y^\dagger] = 1$

The angular momentum of the oscillator is

$$L = x p_y - y p_x.$$

(i) Evaluate L in terms of the a's and $a^\dagger$'s.

(ii) Evaluate $[L, a_x]$ and $[L, a_y]$.

(iii) Give a physical argument that requires $[L, H] = 0$. Use the result from (ii) & check that $[L, H] = 0$

Solution:

(i)

$$p_x = -\frac{i}{2}\sqrt{2m\omega\hbar}\,(a_x - a_x^\dagger)$$

$$x = \frac{1}{2}\sqrt{\frac{2\hbar}{m\omega}}\,(a_x + a_x^\dagger)$$

$$L = x p_y - y p_x = -\frac{i}{4} 2\hbar \left[(a_x + a_x^\dagger)(a_y - a_y^\dagger) - (a_y + a_y^\dagger)(a_x - a_x^\dagger) \right]$$

$$= i\hbar \left[a_x a_y^\dagger - a_y a_x^\dagger \right]$$

A. Mueller

$$[L, a_x] = i\hbar[-a_y a_x^\dagger, a_x] = i\hbar a_y$$

$$[L, a_y] = i\hbar[a_x a_y^\dagger, a_y] = -i\hbar a_x$$

(iii) The Hamiltonian is invariant under rotations in the x,y plane, therefore $[L,H]=0$ since L is the generator of such rotations.

$$[L,H] = [i\hbar(a_x a_y^\dagger - a_y a_x^\dagger), \hbar\omega(a_x^\dagger a_x + a_y^\dagger a_y + 1)]$$

$$= i\hbar^2\omega[a_x a_y^\dagger - a_y a_x^\dagger, a_x^\dagger a_x + a_y^\dagger a_y]$$

$$= i\hbar^2\omega\{\underbrace{[a_x, a_x^\dagger a_x]}_{a_x} a_y^\dagger + a_x \underbrace{[a_y^\dagger, a_y^\dagger a_y]}_{-a_y^\dagger} - a_y \underbrace{[a_x^\dagger, a_x^\dagger a_x]}_{-a_x^\dagger} - a_x^\dagger \underbrace{[a_y, a_y^\dagger a_y]}_{a_y}$$

$$= 0$$

Christ

Suggested Solutions

1. (a) Introduce ϕ to represent rotation of the moving sphere about its center of mass. Equating the total kinetic energy with the potential energy lost gives:

$$\begin{aligned}(R+r)(1-\cos(\theta))mg &= m(r\dot{\phi})^2 + \frac{2}{5}mr^2\dot{\phi}^2 \\ &= \frac{7}{5}mr^2\dot{\phi}^2 \\ &= \frac{7}{5}mv_{\mathrm{cm}}^2\end{aligned}$$

Thus, $v_{\mathrm{cm}}(\theta) = \sqrt{\frac{5}{7}(R+r)(1-\cos(\theta))g}$.

(b) The sphere will fly off when $mv_{\mathrm{cm}}^2/(R+r) > mg\cos(\theta)$ or

$$\frac{5}{7}(1-\cos(\theta)) > \cos(\theta)$$

or

$$\cos(\theta) = 5/13$$

(c) Start with the equation of motion obtained by equating torque and rate of change of angular momentum around the point of contact: $\frac{7}{5}mr^2\ddot{\phi} = mgr\sin\theta$. Relate θ and ϕ by computing the velocity of the moving sphere's center of mass two ways:

$$(R+r)\dot{\theta} = r\dot{\phi}$$

Combining these equations:

$$\ddot{\theta} = \frac{5g}{7(R+r)}\theta$$

or

$$\theta(t) = \frac{\dot{\theta}(0)}{\omega}\sinh(\omega t)$$

Section 3

Question #4

where $\omega = \sqrt{\frac{5g}{7(R+r)}}$.

2. (a) $\psi(\theta) = \frac{1}{\sqrt{2\pi}}e^{ik_n\theta}$, $E_n = \frac{(\hbar k_n)^2}{2m}$ where $k_n = n/R$ and n is an integer.

(b) The same form as in a) except $k_n = n/R + qBR/2$ where n is an integer.

(c) $\psi(\theta_1,\theta_2) = e^{i(\theta_1+\theta_2)P_n/2}\sin[(\theta_1-\theta_2)k_l]$, $E_n = \frac{(\hbar P_n)^2}{4m} + \frac{(\hbar k_l)^2}{m}$, where $P_n = n/R$, $k_l = l/(4R)$ and n, l are integers.

(d) $\psi(\theta_1,\theta_2) = e^{i(\theta_1+\theta_2)P_n/2}e^{i(\theta_1-\theta_2)k_l}$, $E_n = \frac{(\hbar P_n)^2}{4m} + \frac{(\hbar k_l)^2}{m}$, where $P_n = n/R$, $k_l = l/(2R) + \delta/(2\pi R)$ and n, l are integers.

Kim Question #5 ①
section 3

QM: Hydrogen (like) Atom

(a) Find the normalization constant first,

$$1 = \langle\psi|\psi\rangle = A^2[1^2+2^2+2^2+1^2]$$

or $\boxed{A = \frac{1}{\sqrt{10}}}$.

Then

$$\langle E\rangle = \langle\psi|H|\psi\rangle = \frac{1}{10}[\underbrace{E_{100}}_{=E_1}+4\underbrace{E_{210}}_{=E_2}+4E_2+E_2]$$

$$= \frac{1}{10}(E_1+9\underbrace{E_2}_{=\frac{E_1}{2^2}}) = \frac{13}{40}E_1$$

$E_1 = 13.6eV$.

$$\langle L_z\rangle = \langle\psi|L_z|\psi\rangle = \frac{1}{10}[\hbar\cdot 0+4\hbar\cdot 0+4\hbar-\hbar]$$
$$= \frac{3\hbar}{10}$$

$$\langle L^2\rangle = \langle\psi|L^2|\psi\rangle = \frac{1}{10}[0+9\hbar^2\cdot 1\cdot(1+1)]$$
$$= \frac{9\hbar^2}{5}.$$

(b)

$$\Delta H = +m_e g z$$

Then

$$\Delta E = \langle\psi|m_e g z|\psi\rangle$$

up to the first order of $m_e g$.

Note that $[L_z, z] = 0$ & $\langle \psi_{n\ell m} | z | \psi_{n\ell m} \rangle = 0$

Therefore only non-zero contribution in $\langle \psi | z | \psi \rangle$

is the terms $\langle \psi_{210} | z | \psi_{100} \rangle$ and $\langle \psi_{100} | z | \psi_{210} \rangle$

Thus

$$\Delta E = m_e g A^2 \left[2 \langle \psi_{210} | z | \psi_{100} \rangle + 2 \langle \psi_{100} | z | \psi_{210} \rangle \right]$$

$$= \frac{m_e g}{10} \cdot 2 \cdot 2 \cdot \int_{-\pi}^{\pi} d\theta \sin\theta \int_0^{2\pi} d\varphi \int_0^{\infty} r^2 dr \, \frac{2}{2^{3/2} a_0^3} \frac{1}{4\pi}$$

$$\times \left(\frac{r}{a_0}\right) e^{-\frac{3r}{2a_0}} \cos\theta \cdot \underbrace{r\cos\theta}_{= z}$$

$$= m_e g \frac{4}{10} \frac{1}{\sqrt{2}} \frac{1}{a_0^3} \frac{1}{4\pi} a_0^4 \underbrace{\int_{-\pi}^{\pi} d\theta \sin\theta \cos^2\theta}_{2/3} \cdot 2\pi \int_0^{\infty} dt \, t^4 e^{-\frac{3}{2}t}$$

$$= m_e g \frac{a_0}{\sqrt{2} \cdot 5} \left(\frac{2}{3}\right)^5 \underbrace{\int_0^{\infty} ds \, s^4 e^{-s}}_{= 4!}$$

$$= 0.447 \, m_e g.$$

(c)

$$H_{SO} = f(r) \vec{L} \cdot \vec{S} \qquad f(r) : \text{radial function only}$$

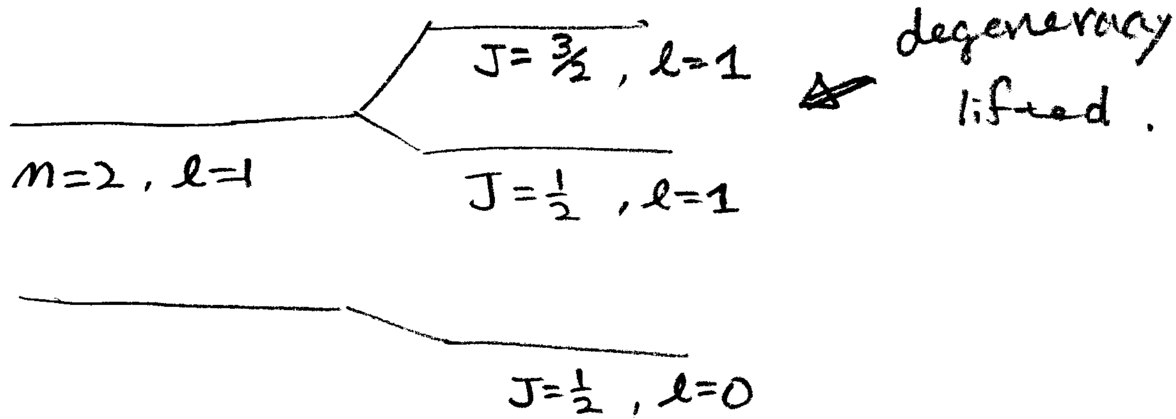
$$= f(r) \frac{1}{2} \left[\vec{J}^2 - \vec{L}^2 - \vec{S}^2 \right]$$

where $\vec{J} = \vec{L} + \vec{S}$

For $l = 0 \Rightarrow j = \frac{1}{2}$ only

For $l = 1 \Longrightarrow j = \frac{3}{2}$ & $\frac{1}{2}$

Thus



Note that we don't have ψ_{200} component in ψ.

Therefore $\underline{\underline{3}}$ different values of measurable energy for the wavefunction ψ.

Hailey Section 3 Question #6

Soln a.) The Hamiltonian is just

$$\frac{L_{op}^2}{2I}\psi = \varepsilon\psi \qquad L_{op}^2 = L_x^2 + L_y^2 + L_z^2$$

$$\psi \sim Y_{\ell m}(\theta\phi) \qquad L_{op}^2 Y_{\ell m} = \ell(\ell+1)\hbar^2$$

$$\varepsilon_{\ell m} = \frac{\ell(\ell+1)\hbar^2}{2I}, \quad \psi \sim Y_{\ell m}(\theta\phi) \quad \varepsilon_{grd} = 0 \quad \ell = 0$$

b.) perturbation theory

$$\varepsilon_{grd}^{(1)} = \langle \ell m | V | \ell m \rangle$$

$V = -pE\cos\theta = -\vec{p}\cdot\vec{E}$ for E along the z-axis

$$\varepsilon_{grd}^{(1)} = \langle \ell m | -pE\cos\theta | \ell m \rangle$$

This matrix element is 0 by parity.

$$\varepsilon_{grd}^{(2)} = \sum_{\ell m} \frac{|\langle \ell m | V | 00 \rangle|^2}{E_{00} - \varepsilon_{\ell m}}$$

$V = -pE\cos\theta = -pE\sqrt{\frac{4\pi}{3}}\,Y_{10}$ since $Y_{10} = \sqrt{\frac{3}{4\pi}}\cos\theta$. The matrix element is

$\sim \langle \ell m | Y_{10} | 00 \rangle \sim \langle \ell m | Y_{10} \rangle$ since

$\langle \theta\phi | 00 \rangle = \frac{1}{\sqrt{4\pi}}$ so only $\ell = 1$ $m = 0$ connects ground state to higher states

$$\langle \ell m | V | 00 \rangle = -pE\sqrt{\frac{4\pi}{3}} \frac{\langle \ell m | 10 \rangle}{\sqrt{4\pi}} = -pE\sqrt{\frac{4\pi}{3}}$$

ie $\int Y_{\ell m}\left(-pE\sqrt{\frac{4\pi}{3}}\, Y_{10}\right)\left(\frac{1}{\sqrt{4\pi}}\right) d\Omega = -\frac{pE}{\sqrt{3}} \underbrace{\int Y_{\ell m} Y_{10}\, d\Omega}_{\delta_{\ell 1}\delta_{m 1}}$

$$= -\frac{pE}{\sqrt{3}}$$

$$E_0 - \mathcal{E}_{\ell m} = \mathcal{E}_0 - \mathcal{E}_{10} = 0 - \frac{\ell(\ell+1)\hbar^2}{2I} = -\frac{\hbar^2}{I}$$

$$\mathcal{E}^{(2)}_{grd} = \frac{(-pE/\sqrt{3})^2}{-\hbar^2/I} = -\frac{p^2E^2I}{3\hbar^2}$$

well maybe they remember that

$$\psi^{(1)}_n = \psi^{(0)}_n + \sum \frac{\langle \psi^{(0)}_n | V | \psi^{(0)}_k \rangle}{E^0_n - E^0_k} \psi^{(0)}_k$$

$$\psi^{(1)}_0 = Y_{00} - \frac{pE/\sqrt{3}}{-\hbar^2/I} Y_{10}$$

$$\psi^{(1)}_0 = Y_{00} + \frac{pEI}{\sqrt{3}\hbar^2} Y_{10}$$

c.) This is constrained rotator

$$-\frac{\hbar^2}{2I}\frac{\partial^2}{\partial\phi^2}\psi = \mathcal{E}\psi \qquad \psi = e^{im\phi}$$

$$\psi(\phi+2\pi) = \pm\psi(\phi)$$

$$\frac{\hbar^2 m^2}{2I} = \mathcal{E}_m \qquad \Rightarrow m = 0, \pm 1, \pm 2 \ldots$$

$$e^{im\phi} \sim \psi$$

d.) $\frac{L_z^2}{2I} - pE\cos\phi = \varepsilon \psi$

$$-\frac{\hbar^2}{2I}\frac{\partial^2\psi}{\partial\phi^2} - pE(1-\phi^2/2)\psi = \varepsilon\psi$$

This is just the simple harmonic oscillator. They know the grd. state wave function is just $\psi \sim e^{-\alpha\phi^2}$

plugging this in you get

$$\left(pE/2 - \frac{2\alpha^2\hbar^2}{I}\right)\phi^2 + \alpha\frac{\hbar^2}{I} - pE = \varepsilon$$

$$\Rightarrow \alpha = \frac{\sqrt{pEI}}{2\hbar} \qquad \varepsilon = \frac{\hbar}{2}\sqrt{\frac{pE}{I}} - pE$$

$$\psi \sim e^{-\sqrt{pEI}/2\hbar\,\phi^2}$$

Columbia University
Department of Physics
QUALIFYING EXAMINATION
January 14, 2004
11:10 AM – 1:10 PM

Modern Physics
Section 4. Relativity and Applied Quantum Mechanics

Two hours are permitted for the completion of this section of the examination. Choose 4 problems out of the 5 included in this section. (You will not earn extra credit by doing an additional problem). Apportion your time carefully.

Use separate answer booklet(s) for each question. Clearly mark on the answer booklet(s) which question you are answering (e.g., Section 4 (Relativity and Applied QM), Question 2; Section 4(Relativity and Applied QM) Question 3, etc.)

Do **NOT** write your name on your answer booklets. Instead clearly indicate your **Exam Letter Code**

You may refer to the single note sheet on 8 ½ x 11" paper (double-sided) you have prepared on Modern Physics. The note sheet cannot leave the exam room once the exam has begun. This note sheet must be handed in at the end of today's exam. Please include your Exam Letter Code on your note sheet. No other extraneous papers or books are permitted.

Simple calculators are permitted. However, the use of calculators for storing and/or recovering formulae or constants is NOT permitted.

Questions should be directed to the proctor.

Good luck!!

Problem 4

Two particles of mass m_1 and m_2 are observed in a certain frame to have four-vectors p_1 and p_2, respectively,

a) Denote the magnitude of three-momentum each particle has in their center-of-mass frame as k. Find an expression entirely in terms of relativistic invariants for k.
b) The relative velocity between the two particles in the center-of-mass frame can also be expressed solely in terms of invariant quantities. Do so.

Hint: You may find it useful to note that the four-vector $(p_1 + p_2)/M$, where $M^2 = (p_1 + p_2)^2$ has a very simple form in the center-of-mass.

Problem 5

In neutral B-mesons, weak and mass eigenstates are not the same. These mesons therefore exhibit particle-antiparticle mixing. Because of this phenomenon, particles produced as pure B^0 or $\overline{B}^0$ weak eigenstates will evolve in time as a superposition of the two states:

$$\left|B^0(t)\right\rangle = a(t)\left|B^0\right\rangle + b(t)\left|\overline{B}^0\right\rangle$$
$$\left|\overline{B}^0(t)\right\rangle = a'(t)\left|B^0\right\rangle + b'(t)\left|\overline{B}^0\right\rangle$$

where $\left|B^0\right\rangle$ and $\left|\overline{B}^0\right\rangle$ are pure B^0 and $\overline{B}^0$ states.

The time evolution of these states is described by 2×2 hermitian mass (M) and decay (Γ) matrices.

$$i\frac{\partial}{\partial t}\begin{pmatrix} a(t) \\ b(t) \end{pmatrix} = \left(\mathbf{M} - \frac{i}{2}\Gamma\right)\begin{pmatrix} a(t) \\ b(t) \end{pmatrix}$$

CPT invariance requires that $M_{11} = M_{22} \equiv M$ and $\Gamma_{11} = \Gamma_{22} \equiv \Gamma$.

a) Calculate the mass and decay width differences between the two mass eigenstates of the B^0 - $\overline{B}^0$ system in terms of the elements of $\boldsymbol{M}$ and Γ. You may assume that CP-violation is negligible, *i.e.* that the phase difference between M_{12} and Γ_{12} is zero.
b) What are the functions $a(t)$, $a'(t)$, $b(t)$, $b'(t)$ in terms of the mass and decay matrix elements?

Section 4 Question #1

Hailey

Let K = rest frame

$\rho(\theta\phi)$ = # stars / solid Angle

K' = moving frame $\rho(\theta'\phi')$ = # stars / solid Angle

Note $\rho(\theta\phi)\,d\Omega = \rho(\theta'\phi')\,d\Omega'$

since the number of stars they see is countable and thus an invariant

$$\rho(\theta'\phi') = \rho(\theta\phi)\frac{d\Omega}{d\Omega'} = \frac{N}{4\pi}\frac{d\Omega}{d\Omega'}$$

$$\rho(\theta'\phi') = \frac{N}{4\pi}\frac{d\cos\theta}{d\cos\theta'} \quad \text{since } d\phi' = d\phi$$

$$\cos\theta = \frac{\cos\theta' + \beta}{1+\beta\cos\theta'}$$ $\Leftarrow$ they can derive if they don't know

$$\frac{d\cos\theta}{d\cos\theta'} = \frac{1-\beta^2}{(1+\beta\cos\theta')^2}$$

$$\Rightarrow \rho(\theta'\phi') = \frac{N}{4\pi}\frac{(1-\beta^2)}{(1+\beta\cos\theta')^2}$$

This has a maximum at $\theta' = \pi$

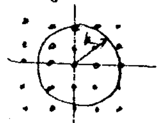
so the stars bunch up in the forward direction.

Heinz Section 4, Question 3

Heinz - Problem 1

(a) $E(k_x, k_y, n) = \frac{\hbar^2}{2m}\left[k_x^2 + k_y^2 + \left(\frac{n\pi}{\ell}\right)^2\right]$, $n = 1, 2, 3, \ldots$

(b) First find ρ_E^{2D} for free electrons by the usual state counting. Assume a lateral dimension of $L \times L$. Then, for periodic boundary conditions, values of k_x, k_y are spaced by $\Delta k = 2\pi/L$



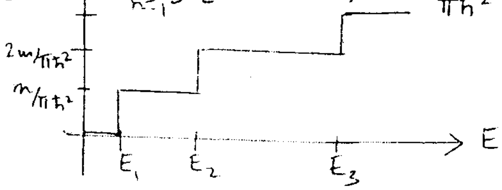
Number of states up to energy E

$$N = \underbrace{2}_{\text{spin deg. for } s=\frac{1}{2}} \frac{A}{(\Delta k)^2} = \frac{2\pi k^2}{(2\pi/L)^2} = \frac{mL^2}{\pi\hbar^2}E$$

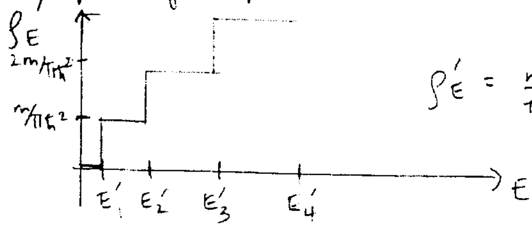
$$\Rightarrow \rho_E^{2D} = \frac{1}{L^2}\frac{dN}{dE} = \frac{m}{\pi\hbar^2}\theta(E)$$

Now include the z-axis quantization

$$\rho_E = \sum_{n=1}^{\infty} \rho_E^{2D}(E - E_n) = \frac{m}{\pi\hbar^2}\sum_n \theta(E - E_n), \quad E_n = \frac{\hbar^2\pi^2}{2mL^2}n^2$$



(c) The only effect of a finite well is to lower the values of E_n'



$$\rho_E' = \frac{m}{\pi\hbar^2}\sum_n \theta(E - E_n')$$

(d) $$n_s = \int_0^\infty \rho_E(E) \underbrace{f(E)}_{\text{F-D distribution.}} dE = \int_0^\infty \rho_E(E)\left[e^{(E-\mu)/kT} + 1\right]^{-1} dE \quad (*)$$

This eqn implicitly defines the chemical potential μ.

Then $E_{1/3}$ is given by $f(E_{1/3}) = \left[e^{(E_{1/3}-\mu)/kT} + 1\right]^{-1} = 1/3$

or $\| \; E_{1/3} = kT\ln 2 + \mu$ with μ defined by $(*)$.

Quals 2004
Section 4 – Modern Physics: Relativity (Four Vector Rel Vel)
Question # ~~5~~ 4
Bill Zajc
12/29/03

Two particles of mass m_1 and m_2 are observed in a certain frame to have four-vectors p_1 and p_2, respectively,

a) Denote the magnitude of three-momentum each particle has in their center-of-mass frame as k. Find an expression entirely in terms of relativistic invariants for k.
b) The relative velocity between the two particles in the center-of-mass frame can also be expressed solely in terms of invariant quantities. Do so.

Hint: You may find it useful to note that the four-vector $(p_1 + p_2)/M$, where $M^2 = (p_1 + p_2)^2$, has a very simple form in the center-of-mass.

NO SOLUTION PROVIDED

Section 4, Question 6
Evans

Qual's – 2004: Quantum Solutions (B-Mixing)

(H. Evans)

Part a)

First find the eigenvalues ($\lambda_\pm$) and eigenvectors ($\mathbf{x}_\pm$) of the mass/decay matrix:

$$\mathbf{D} \equiv \left(\mathbf{M} - \frac{i}{2}\Gamma \right)$$

Eigenvalues:

$$\mathbf{Dx} = \lambda \mathbf{x} \Rightarrow \begin{vmatrix} d-\lambda & d_{12} \\ d_{21} & d-\lambda \end{vmatrix} = 0 \Rightarrow \lambda^2 - 2d\lambda + (d^2 - d_{12}d_{21}) = 0$$

$$\lambda_\pm = d \pm \sqrt{d_{12}d_{21}} = \left(M - \frac{i}{2}\Gamma \right) \pm \sqrt{\left(M_{12} - \frac{i}{2}\Gamma_{12} \right)\left(M_{21} - \frac{i}{2}\Gamma_{21} \right)}$$

$$= \left(M - \frac{i}{2}\Gamma \right) \pm \sqrt{\left(M_{12} - \frac{i}{2}\Gamma_{12} \right)\left(M_{12}^* - \frac{i}{2}\Gamma_{12}^* \right)}$$

since M_{12} and Γ_{12} have the same phase, ϕ:

$$\lambda_\pm = \left(M - \frac{i}{2}\Gamma \right) \pm \left(|M_{12}| - \frac{i}{2}|\Gamma_{12}| \right)$$

The time evolution equation for the mass eigenstates is then:

$$i\frac{\partial}{\partial t}\mathbf{x}_\pm(t) = \lambda_\pm \mathbf{x}_\pm(t) \Rightarrow \mathbf{x}_\pm(t) \propto exp(-i\lambda_\pm t) = exp\left(-iM_\pm t - \frac{\Gamma_\pm}{2}t \right)$$

where $M_\pm = Re(\lambda_\pm)$ and $\Gamma_\pm = Im(\lambda_\pm)$.
The mass and width differences are then:

$$\Delta m \equiv M_+ - M_- = 2|M_{12}|$$

$$\Delta\Gamma \equiv \Gamma_+ - \Gamma_- = 2|\Gamma_{12}|$$

Part b)

Eigenvectors:

$$\mathbf{D}\mathbf{x}_\pm = \lambda_\pm \mathbf{x}_\pm$$
$$\Rightarrow dx_{\pm 1} + d_{12}x_{\pm 2} = \lambda_\pm x_{\pm 1} \text{ and } d_{21}x_{\pm 1} + dx_{\pm 2} = \lambda_\pm x_{\pm 2}$$
$$\Rightarrow x_{\pm 2} = \pm x_{\pm 1}\sqrt{\frac{d_{21}}{d_{12}}}$$

If we defined the weak basis as:

$$\left|B^0\right\rangle \propto \begin{pmatrix}1\\0\end{pmatrix} \text{ and } \left|\overline{B}^0\right\rangle \propto \begin{pmatrix}0\\1\end{pmatrix}$$

Then the mass eigenstates are:

$$\left|B_\pm\right\rangle = p\left|B^0\right\rangle \pm q\left|\overline{B}^0\right\rangle$$

with

$$\frac{q}{p} = \sqrt{\frac{d_{21}}{d_{12}}} = \sqrt{\frac{M_{12}^* - i\Gamma_{12}^*/2}{M_{12} - i\Gamma_{12}/2}}$$

These mass eigenstates evolve separately as:

$$\left|B_\pm(t)\right\rangle = exp(-i\lambda_\pm t)\left|B_\pm\right\rangle$$

Writing the weak eigenstates in terms of the mass eigenstates:

$$\left|B^0\right\rangle = \frac{1}{2p}\left(\left|B_+\right\rangle + \left|B_-\right\rangle\right) \qquad \text{and} \qquad \left|\overline{B}^0\right\rangle = \frac{1}{2q}\left(\left|B_+\right\rangle - \left|B_-\right\rangle\right)$$

The the mass eigenstates time evolution become:

$$p\left|B^0(t)\right\rangle + q\left|\overline{B}^0(t)\right\rangle = exp(-i\lambda_+ t)\left[p\left|B^0\right\rangle + q\left|\overline{B}^0\right\rangle\right]$$
$$p\left|B^0(t)\right\rangle - q\left|\overline{B}^0(t)\right\rangle = exp(-i\lambda_- t)\left[p\left|B^0\right\rangle - q\left|\overline{B}^0\right\rangle\right]$$

Solving these simultaneous equations gives:

$$\left|B^0(t)\right\rangle = g_+(t)\left|B^0\right\rangle + \frac{q}{p}g_-(t)\left|\overline{B}^0\right\rangle$$

$$\left|\overline{B}^0(t)\right\rangle = g_+(t)\left|\overline{B}^0\right\rangle + \frac{p}{q}g_-(t)\left|B^0\right\rangle$$

where the time evolution coefficients are:

$$g_\pm(t) = \frac{1}{2}\left[exp(-i\lambda_+ t) \pm exp(i\lambda_- t)\right]$$