Columbia University
Department of Physics
QUALIFYING EXAMINATION
January 12, 2004
9:00 AM – 11:00 AM

Classical Physics
Section 1. Classical Mechanics

Two hours are permitted for the completion of this section of the examination. Choose <u>4 problems</u> out of the 5 included in this section. (You will <u>not</u> earn extra credit by doing an additional problem). Apportion your time carefully.

Use separate answer booklet(s) for each question. Clearly mark on the answer booklet(s) which question you are answering (e.g., Section 1 (Classical Mechanics), Question 1; Section 1(Classical Mechanics) Question 3, etc.)

Do **NOT** write your name on your answer booklets. Instead clearly indicate your **Exam Letter Code**

You may refer to the single note sheet on 8 ½ x 11" paper (double-sided) you have prepared on Classical Physics. The note sheet cannot leave the exam room once the exam has begun. This note sheet must be handed in at the end of today's exam. Please include your Exam Letter Code on your note sheet. No other extraneous papers or books are permitted.

Simple calculators are permitted. However, the use of calculators for storing and/or recovering formulae or constants is NOT permitted.

Questions should be directed to the proctor.

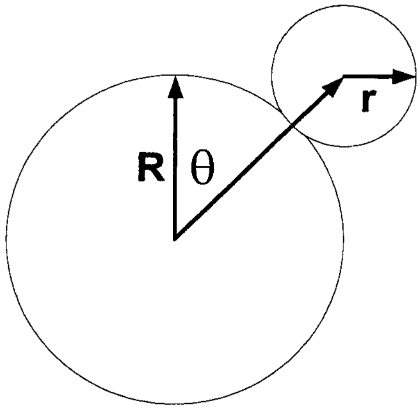
Good luck!!

Problem 1

A simple pendulum consists of a massless rod of length R with a mass m at the bottom. The rod pivots on a hinge that constrains the motion of the pendulum to the vertical plane. The hinge is attached to the lower end of a vertical shaft that is made to rotate with a constant angular frequency ω by a synchronous motor.

a) Write down the Lagrangian and the equation of motion in a stationary coordinate system.
b) What is the solution corresponding to stable small oscillations about the position of minimum energy and what is the condition for this type of motion?
c) What is the frequency of small oscillations?
d) What is the Hamiltonian, H, in the fixed coordinate system? What is the Hamiltonian in a rotating system attached to the shaft of the motor?
e) Are $T+V$ (kinetic + potential energy) and H conserved in each of the coordinate systems mentioned above?

Problem 2

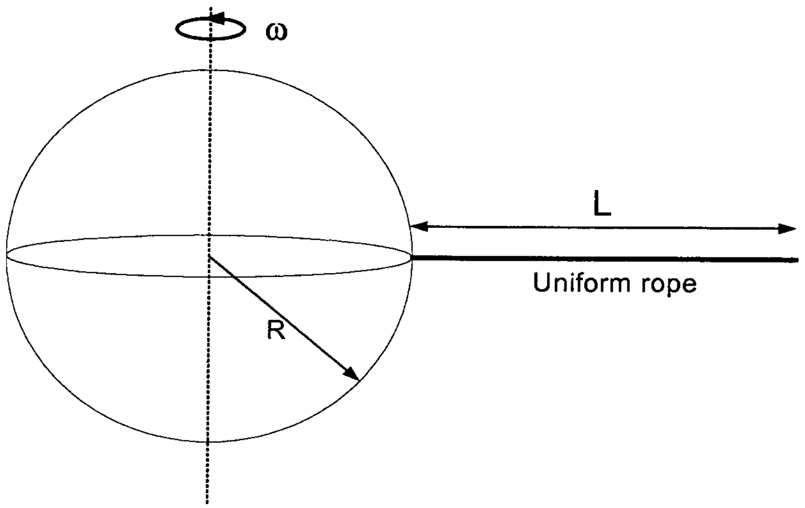


A uniform sphere of radius r and mass m rolls without sliding on the outer surface of a stationary sphere of radius R. The position of the rolling sphere is described by angle, θ, as shown in the figure. If the upper sphere starts from rest at the top of the stationary sphere $(\theta = 0)$,

a) Find the velocity of the center of mass of the moving sphere as a function of θ.
b) Determine that value of θ at which the moving sphere flies off the stationary one.
c) If the moving sphere begins at $t = 0$ with $\theta = 0$ but $\dot{\theta}(0) \neq 0$ find $\theta(t)$ in terms of $\dot{\theta}(0)$ for small t.

Problem 3

A recent Science Times article featured the concept of a "space elevator". This is a free hanging rope in stationary orbit around the earth above the equator. You could send an elevator up this rope to launch objects into space at less cost than required for shuttle flights. Imagine such a rope just reached the earth's surface. Find an expression for the tension in the rope as a function of height, y, off the earth's surface. Assume the rope has length L, and mass m, and that the earth has radius R and mass M and rotates at angular velocity ω. What length, L, allows the rope to hang freely (i.e. without being attached to the earth's surface) ?



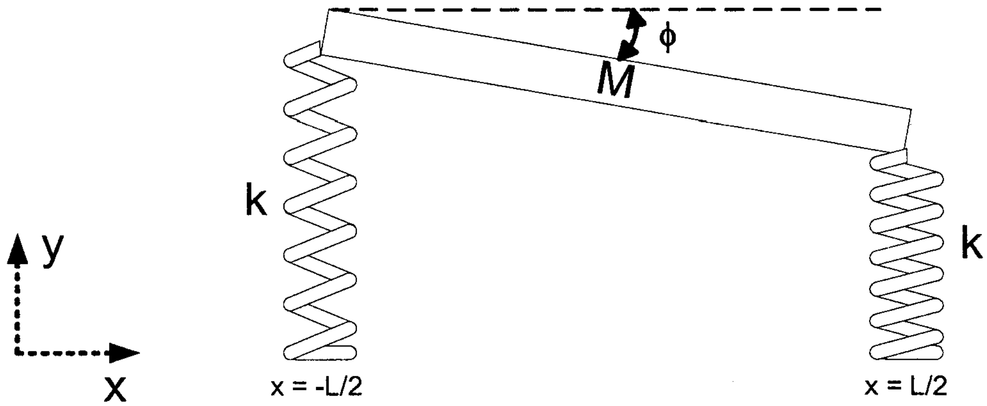
Problem 4

Consider circular orbits in the field of a central force $f(r) = -kr^n$

a) What is the relation between orbital velocity v and radius r?
b) For what values of n are these circular orbits stable to small perturbations?
c) Specifically, are the circular orbits stable or unstable for $n = -3$?

Problem 5

A thin, uniform density, bar of length L and mass M is support by two springs with identical spring constants k and unloaded lengths l. The center of mass of the bar is constrained to move vertically with displacement $y(t)$, but the bar can rotate in the x-y plane with angle $\phi(t)$.



a) Determine the Lagrangian of the system for small displacements taking gravity into account.

b) Solve the Euler-Lagrange equations of motion to determine the frequencies and eigenvectors for the normal modes of the system.

c) If at $t=0$ the end of the bar at $x=L/2$ is depressed by a *small* amount, d, while the other end is held in its equilibrium position (i.e. $y_1(t=0)=l$, $y_2(0)=l-d$) and all initial generalized velocities vanish, find expressions for the time dependence of the subsequent vertical displacement and angular rotation of the bar, $y(t)$ and $\phi(t)$.

Email

Delivered-To: lalla@phys.columbia.edu
Date: Mon, 22 Dec 2003 02:01:46 -0500
From: Elena Aprile <age@astro.columbia.edu>
Subject: Aprile_Quals2004_Mechanics Problem
To: cole@nevis1.columbia.edu
Cc: lalla@phys.columbia.edu
X-Mailer: Microsoft Outlook IMO, Build 9.0.2416 (9.0.2910.0)
Importance: Normal

Section 1-:Classical Mechanics
Question # ~~5~~ 1

Hi Brian, I attach a problem for the quals; sorry for being late. Elena

MECHANICS_PROBLEM

A simple pendulum consists of a massless rod of length R with a mass m at the bottom. The rod pivots on a hinge that constrains the motion of the pendulum to the vertical plane. The hinge is attached to the lower end of a vertical shaft that is made to rotate with a constant angular frequency w by a synchronous motor.

(1) Write down the Lagrangian and the equation of motion in a stationary coordinate system.

(2) What is the solution corresponding to stable small oscillations about the position of minimum energy and what is the condition for this type of motion? What is the frequency of small oscillations?

(3) What torque does the motor exert when the pendulum oscillates as above.

(4) What is the Hamiltonian in the fixed coordinate system? What is the Hamiltonian in a rotating system attached to the shaft of the motor?

(5) Are T+V and H conserved in each of the coordinate systems mentioned above?

NO SOLUTION

Christ Section 1, Question # 2

Suggested Solutions

1. (a) Introduce ϕ to represent rotation of the moving sphere about its center of mass. Equating the total kinetic energy with the potential energy lost gives:

$$\begin{aligned}(R+r)(1-\cos(\theta))mg &= m(r\dot{\phi})^2 + \frac{2}{5}mr^2\dot{\phi}^2 \\ &= \frac{7}{5}mr^2\dot{\phi}^2 \\ &= \frac{7}{5}mv_{\rm cm}^2\end{aligned}$$

Thus, $v_{\rm cm}(\theta) = \sqrt{\frac{5}{7}(R+r)(1-\cos(\theta))g}$.

(b) The sphere will fly off when $mv_{\rm cm}^2/(R+r) > mg\cos(\theta)$ or

$$\frac{5}{7}(1-\cos(\theta)) > \cos(\theta)$$

or

$$\cos(\theta) = 5/13$$

(c) Start with the equation of motion obtained by equating torque and rate of change of angular momentum around the point of contact: $\frac{7}{5}mr^2\ddot{\phi} = mgr\sin\theta$. Relate θ and ϕ by computing the velocity of the moving sphere's center of mass two ways:

$$(R+r)\dot{\theta} = r\dot{\phi}$$

Combining these equations:

$$\ddot{\theta} = \frac{5g}{7(R+r)}\theta$$

or

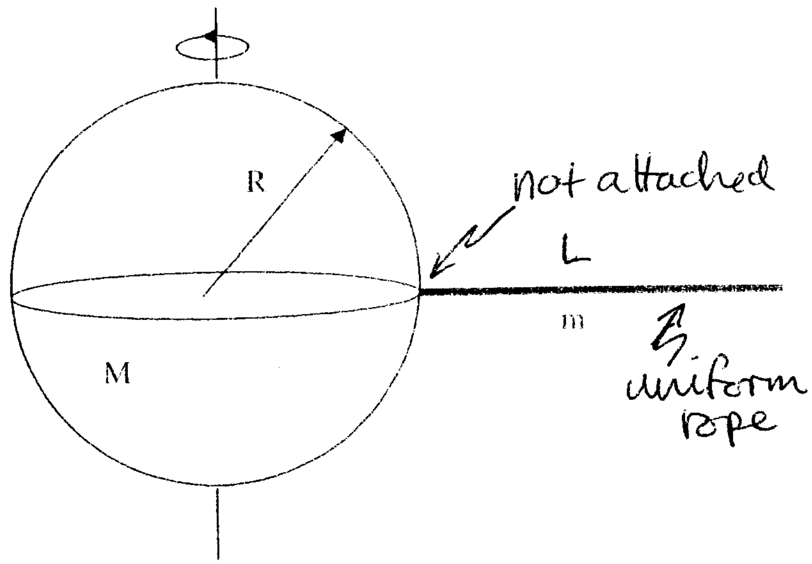
$$\theta(t) = \frac{\dot{\theta}(0)}{\omega}\sinh(\omega t)$$

where $\omega = \sqrt{\frac{5g}{7(R+r)}}$.

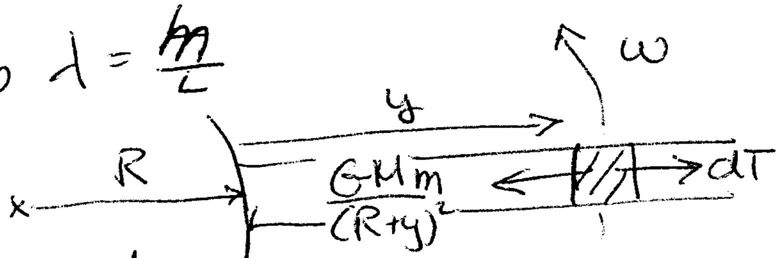
Section 1, Question #63

Mechanics Turtz

Q3. A recent Science Times article featured the concept of a space elevator. This is a "free hanging" rope in stationary orbit around the equator. You could then send an elevator up this rope to launch objects into space at less cost than the shuttle. Imagine that the rope just reached the earth's surface. What is the expression for the tension in the rope? How long does the rope have to be? Assume the rope has length L, mass m, and the earth has radius R and mass M.



A. The density is $\lambda = \frac{m}{L}$



$$\frac{GMm}{(R+y)^2} + dT = m\omega^2(R+y)$$

$$\therefore dT = \lambda dy\,\omega^2(R+y) - \frac{GM\lambda dy}{(R+y)^2}$$

$$T(y) = \frac{GM\lambda}{(R+y)} + \frac{\lambda\omega^2}{2}(R+y)^2 \Big|_y^L$$

$$\therefore T(y) = \frac{GM\lambda}{R+L} - \frac{GM\lambda}{R+y} + \frac{\lambda\omega^2(R+L)^2}{2} - \frac{\lambda\omega^2(R+y)^2}{2}$$

$$= GM\lambda\left(\frac{1}{R+L} - \frac{1}{R+y}\right) + \frac{\lambda\omega^2}{2}\left[R^2 + 2LR + L^2 - R^2 - 2Ry - y^2\right]$$

$$= GM\lambda\left(\frac{y-L}{(R+L)(R+y)}\right) + \frac{\lambda\omega^2}{2}\left(2R(L-y) + L^2 - y^2\right)$$

$$\boxed{T(y) = (L-y)\lambda\left[\frac{-GM}{(R+L)(R+y)} + \frac{\omega^2}{2}(2R+L+y)\right]}$$

So the length must be such that $T = 0$ at $y = 0$

$$\therefore \frac{GM}{(R+L)R} = \frac{\omega^2}{2}(2R+L)$$

$$\therefore \frac{2GM}{\omega^2 R} = (2R+L)(R+L) = 2R^2 + 3RL + L^2$$

$$\therefore L^2 + (3R)L + \left[2R^2 - \frac{2GM}{\omega^2 R}\right] = 0$$

$$\therefore L = \frac{-3R \pm \sqrt{9R^2 - 8R^2 - \frac{8GM}{\omega^2 R}}}{2}$$

$$\boxed{L = -\frac{3}{2}R \pm \frac{1}{2}\sqrt{R^2 - \frac{8GM}{\omega^2 R}}}$$

Section 1, Question #4

Problem ~~2~~ 4 (Mechanics - Central Force)

(M.Shaevitz)

For the following questions assume that there is a central force with the form ($k > 0$)

$$f(r) = -kr^n$$

a) For circular orbits, what is the relation between velocity, v, the radius, r, and n.

b) For what values of n are these circular orbits stable to small perturbations?

c) Specifically, are the circular orbits stable or unstable for $n = -3$?

Solution:

a) From the radial equation of motion

$$m\,\ddot{r} = \frac{mv^2}{r} + f(r)$$

a circular orbit will have $\ddot{r} = 0$ giving (where L = angular momentum)

$$\frac{mv^2}{r} = kr^n = \frac{L^2}{mr^3}$$

$$L = mrv$$

$$v = \sqrt{\frac{k}{m}r^{n+1}}$$

b) For a circular orbit with radius equal a, let $r = a + x$, then

$$m\,\ddot{x} = \frac{L^2}{m(a+x)^3} + f(a+x)$$

Expanding as a power series in $\frac{x}{a}$ give

$$m\,\ddot{x} = \frac{L^2}{ma^3}\left(1 - 3\frac{x}{a} + \ldots\right) + \left(f(a) + f'(a)x + \ldots\right)$$

Keeping only the leading terms and using the conditions for a circular orbit then leads to

$$m\,\ddot{x} + \left(\frac{-3}{a}f(a) - f'(a)\right)x = 0$$

For stable orbits, coeficient of x must be greater than zero

$$-\left(f(a) + \frac{a}{3}f'(a)\right) > 0$$

Given the power law force $f(r) = -kr^n$, the stability condition then becomes

$$-ka^n - \frac{a}{3}kna^{n-1} < 0$$

or

$$n > -3$$

c) For $f(r) = -kr^{-3}$, the above equations becomes

$$f(a) + \frac{a}{3}f'(a) = 0$$

$$m\,\ddot{x} = 0$$

This then gives a non-stable orbit where any perturbation is not corrected by a restorative force.

Section 1, Question 1 # 5

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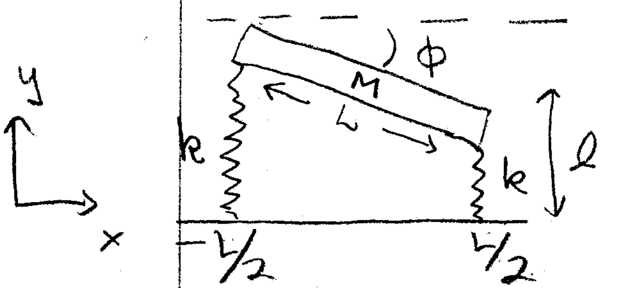


A thin uniform density bar of length L and mass M is supported by two springs with (identical) constants k and unloaded lengths ℓ

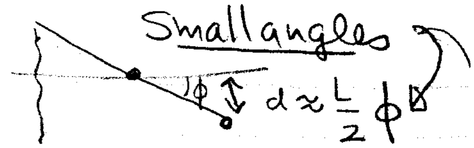
The center of mass of the rod is constrained to move vertically (with $y(t)$), but the bar can rotate with $\phi(t)$ in the xy plane

a) Determin the Lagrangian for small displacements taking gravity g into account

b) Solve the Euler Lagrange equations of motion for the normal modes and frequencies

c) If at $t=0$ the end of the bar at $x = L/2$ is depressed by a small amound d, i.e. $y = \ell - d$ while the other end at $x = -L/2$ is in its equilibrium position $y = \ell$, and if initially all generalized velocities vanish, What is the subsequent $y(t)$ and $\phi(t)$?

Small angles



Solution:

1) $T = \frac{1}{2} M \dot{y}^2 + \frac{1}{2} I \dot{\phi}^2$

$$I = \int_{-L/2}^{L/2} dx\, x^2 \frac{M}{L} = 2\, \frac{1}{3}\left(\frac{L}{2}\right)^3 \frac{M}{L} = \frac{1}{12} M L^2$$

$$V = Mgy + \frac{1}{2} k \left\{ \left(y - \ell + \frac{L}{2}\phi\right)^2 + \left(y - \ell - \frac{L}{2}\phi\right)^2 \right\}$$

Equilibrium: $\frac{\partial V}{\partial y} = 0 = \frac{\partial V}{\partial \phi}$

a) $Mg + k\left(\left(y_{eq} - \ell + L/2\,\phi\right) + \left(y_{eq} - \ell - L/2\,\phi\right)\right) = 0$

$$y_{eq} = \ell - Mg/2k$$

b) $\frac{kL}{2}\left[\left(y - \ell + L/2\, \phi_{eq}\right) - \left(y - \ell - L/2\, \phi_{eq}\right)\right] = 0$

$$\phi_{eq} = 0$$

$$V(y,\phi) \cong V_{eq} + \frac{1}{2}(y - y_{eq})^2 \left(\frac{\partial^2 V}{\partial y^2}\right)_{eq} + \frac{1}{2}(\phi - \phi_{eq})^2 \left(\frac{\partial^2 V}{\partial \phi^2}\right)_{eq}$$

$$= V_{eq} + \eta^2 k + \left(\frac{kL^2}{4}\right)\phi^2$$

$\eta = y - y_{eq}$ = small vertical displacement

$$\boxed{L = \frac{1}{2} M \dot{\eta}^2 + \frac{ML^2}{24}\dot{\phi}^2 - k\eta^2 - \frac{kL^2}{4}\phi^2}$$

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C2

b) Euler Lagrange equation $\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} = \frac{\partial L}{\partial q_i}$

1) $M\ddot{\eta} = -2k\eta \Rightarrow \ddot{\eta} + \omega_1^2 \eta = 0$

2) $\left(\frac{ML^2}{12}\right)\ddot{\phi} = -\left(\frac{kL^2}{2}\right)\phi \Rightarrow \ddot{\phi} + \omega_2^2 \phi = 0$

$$\omega_1^2 = \frac{2k}{M} \qquad \omega_2^2 = \frac{6k}{M}$$

$$\omega_2 = \sqrt{3}\,\omega_1$$

$$\begin{cases} \eta(t) = A\cos(\omega_1 t + \delta_1) \\ \phi(t) = B\cos(\omega_2 t + \delta_2) \end{cases}$$

c) End point height displacements (<u>small</u>)

Convention

$$Z_\pm(t) \equiv \eta(t) \mp L/2\,\phi(t)$$

Given $Z_+(0) = -d = \eta(0) - L/2\,\phi(0)$

$Z_-(0) = 0 = \eta(0) + L/2\,\phi(0)$

$\Rightarrow \eta(0) = -d/2\,,\quad \phi(0) = -\frac{2}{L}\eta(0) = \frac{d}{L}$

Also $\dot{Z}_+(0) = 0 = \dot{Z}_-(0)$

$\Rightarrow \dot{\eta}(0) = 0 = \dot{\phi}(0)$

Use these initial conditions to fix $A\ B\ \delta_1\ \delta_2$ from part b:

$$\left.\begin{aligned} A\cos\delta_1 &= -d/2 \\ B\cos\delta_2 &= d/L \\ A\omega_1\sin\delta_1 &= 0 \\ B\omega_2\sin\delta_2 &= 0 \end{aligned}\right\} \Rightarrow \begin{cases} A = -d/2 \\ B = d/L \\ \delta_1 = \delta_2 = 0 \end{cases}$$

Columbia University
Department of Physics
QUALIFYING EXAMINATION
January 12, 2004
11:10 AM – 1:10 PM

Classical Physics
Section 2. Electricity, Magnetism & Electrodynamics

Two hours are permitted for the completion of this section of the examination. Choose 4 problems out of the 5 included in this section. (You will not earn extra credit by doing an additional problem). Apportion your time carefully.

Use separate answer booklet(s) for each question. Clearly mark on the answer booklet(s) which question you are answering (e.g., Section 2 (Electricity etc.), Question 2; Section 2(Electricity etc.) Question 4, etc.)

Do **NOT** write your name on your answer booklets. Instead clearly indicate your **Exam Letter Code**

You may refer to the single note sheet on 8 ½ x 11" paper (double-sided) you have prepared on Classical Physics. The note sheet cannot leave the exam room once the exam has begun. This note sheet must be handed in at the end of today's exam. Please include your Exam Letter Code on your note sheet. No other extraneous papers or books are permitted.

Simple calculators are permitted. However, the use of calculators for storing and/or recovering formulae or constants is NOT permitted.

Questions should be directed to the proctor.

Good luck!!

Problem 1

A point-like electric dipole $\vec{p}$ is embedded at the center of a sphere of radius R. The sphere is made of linear dielectric material with dielectric constant ε_r and is surrounded by vacuum. Determine the electric potential inside and outside the sphere making sure you get the correct answer in the limit $\varepsilon_r \to 1$. *Hint:* the potential at small distances must approach the potential of a dipole in an infinitely large dielectric medium (SI units),

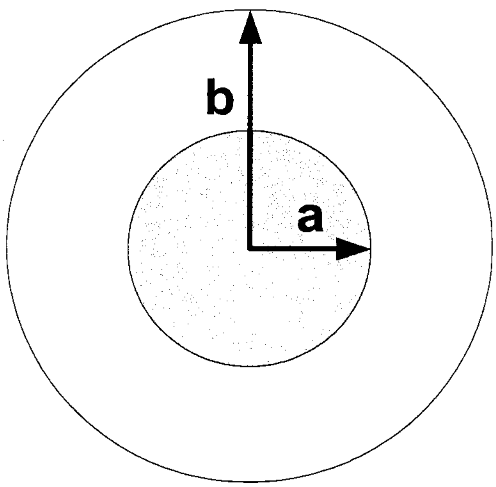
$$\Phi = \frac{1}{4\pi\varepsilon_0\varepsilon_r}\frac{\vec{p}\cdot\vec{r}}{r^3} = \frac{p\cos\theta}{4\pi\varepsilon_0\varepsilon_r r^2}.$$

Problem 2

A coaxial cable consists of two cylindrical conductors. The inner conductor is a solid cylinder of radius a, and the outer conductor is a thin cylindrical shell of radius b. A current I flows in the inner conductor and current $-I$ flows in the outer conductor. Assume that the current in the inner conductor is uniformly distributed across the cross-section of the conductor.

a) Show that the inductance L per unit length l is given by $\frac{L}{l} = \frac{\mu_0}{2\pi}\ln\frac{b}{a} + \frac{\mu_0}{4\pi}$ (you may alternatively give the result in CGS units).

b) What gives rise to the second term in the result in part a? To answer this, consider how the result changes if you assume the inner conductor is a thin cylindrical shell of radius a.



Problem 3

Two halves of a spherical metallic shell of radius R and infinite conductivity are separated by a small insulating gap.

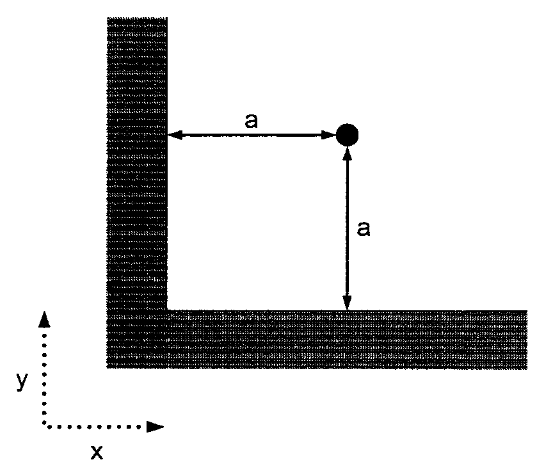
a) If the two shells are maintained at constant potentials $+V$ and $-V$, respectively find the electrostatic potential outside the hemispheres up through and including terms of order $(1/r)^2$.

b) Find the static dipole moment of the system with the conductors held at the potentials given in part a).

c) Now the potentials of the shells are made to vary slowly with time according to $\pm V\cos\omega t$. The electric dipole of the system now varies sinusoidally with time and the system gives off electric dipole radiation. Find the time-averaged power radiated by the two shells as a function of the angle θ (defined with respect to the electric dipole moment) and frequency ω.

Problem 4

A uniformly charged wire with *constant* charge per unit length, $-\lambda$ (charge density $\rho(\vec{r}) = -\lambda\delta(x-a)\delta(y-a)$), runs parallel to and is a distance, a, from the surfaces of two planar, perfect conductors oriented perpendicular to each other. The planes intersect at $x=y=0$. Find the surface charge density, $\sigma(x)$ and $\sigma(y)$ on the surfaces of the horizontal and vertical conducting planes, respectively.



Problem 5

The Proca equations describe electromagnetism with photons that have a non-zero rest mass. The electric and magnetic fields, $\vec{E}$ and $\vec{B}$, are defined in terms of a scalar potential, Φ, and a vector potential, $\vec{A}$, in the usual way, $\vec{B} = \vec{\nabla} \times \vec{A}$ and $\vec{E} = -\vec{\nabla}\Phi - \frac{1}{c}\frac{\partial \vec{A}}{\partial t}$. The four Proca equations involve $\vec{E}$, $\vec{B}$, Φ, and $\vec{A}$ and are a generalization of the four Maxwell's equations:

$$\text{(I) } \vec{\nabla} \cdot \vec{B} = 0 \qquad \text{(I') } \vec{\nabla} \times \vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t}$$

$$\text{(II) } \vec{\nabla} \cdot \vec{E} + \mu^2 \Phi = 4\pi\rho \qquad \text{(II') } \vec{\nabla} \times \vec{B} + \mu^2 \vec{A} = \frac{4\pi}{c}\vec{J} + \frac{1}{c}\frac{\partial E}{\partial t}$$

The first two equations are equivalent to the definitions of $\vec{E}$ and $\vec{B}$ in terms of Φ, and $\vec{A}$. The second two equations involve a new parameter, μ, which has dimensions of reciprocal length and is related to the photon mass via, $\mu = mc/\hbar$. The sources ρ and $\vec{J}$ are the usual charge and current density that satisfy the charge/current conservation law $\vec{\nabla} \cdot \vec{J} + \frac{\partial \rho}{\partial t} = 0$.

a) Show that the Proca equations together with charge/current conservation require that $\vec{A}$ and Φ satisfy the Lorentz condition, $\vec{\nabla} \cdot \vec{A} + \frac{1}{c}\frac{\partial \Phi}{\partial t} = 0$.

b) Using the Proca equations together with the Lorentz condition from part a), find the generalized wave equations that Φ and the Cartesian components of $\vec{A}$ satisfy in terms of the sources ρ and $\vec{J}$.

c) Consider the plane wave solutions, $\begin{Bmatrix} \vec{A}(z,t) \\ \Phi(z,t) \end{Bmatrix} = \mathrm{Re}\left[\begin{Bmatrix} \vec{A}_0 \\ \Phi_0 \end{Bmatrix} e^{i(kz - \omega t)} \right]$

in source-free space ($\rho = 0$ and $\vec{J} = 0$). ω is a given positive constant, while $\vec{A}_0$ and Φ_0 are unspecified constants. Find the constant k in terms of ω, μ, and c. Show that there is a cut-off frequency, ω_c, such that for $\omega > \omega_c$ the wave propagates without attenuation while for $\omega < \omega_c$ the wave does not propagate but is just attenuated in the (positive or negative) z direction. Find the value for ω_c. For $\omega > \omega_c$ find the phase velocity of the given plane wave in terms of ω, μ, and c.

d) For $\omega > \omega_c$, consider a ***longitudinal*** plane wave with the vector potential, $\vec{A}(z,t)$ in the z-direction: $\vec{A}_0 = A_0\hat{z}$, where A_0 is a given positive constant. Find the plane wave's scalar potential $\Phi(z,t)$, magnetic field $\vec{B}(z,t)$, and electric field $\vec{E}(z,t)$ in terms of A_0, ω, μ, and c.

Problem 1 SOLUTION

general solution of Laplace's equation in spherical coordinates

$$\phi = \sum_n \left[A_n r^n P_n(\cos\theta) + \frac{B_n}{r^{n+1}} P_n(\cos\theta) \right]$$

ϕ inside: for $r \to 0$, the potential approaches the potential of a dipole in a large dielectric,

$$\phi = \frac{1}{4\pi\varepsilon} \frac{\vec{p}\,\vec{r}}{r^3} = \frac{1}{4\pi\varepsilon} \frac{p\cos\theta}{r^2},$$

so inside, $B_n = 0$ for $n \neq 1$, but

$$B_1 = \frac{p}{4\pi\varepsilon}$$

$$\Rightarrow \quad \phi_{in} = \frac{p}{4\pi\varepsilon} \frac{1}{r^2} P_1(\cos\theta) + \sum_n A_n r^n P_n(\cos\theta)$$

ϕ outside:

$$\phi_{out} = \sum_n \frac{B_n}{r^{n+1}} P_n(\cos\theta)$$

ϕ is continuous at $r = R$, so $\phi_{in}\big|_{r=R} = \phi_{out}\big|_{r=R}$

$$\Rightarrow \quad \frac{p}{4\pi\varepsilon} \frac{1}{R^2} P_1(\cos\theta) + \sum_n A_n R^n P_n(\cos\theta) = \sum_n \frac{B_n}{R^{n+1}} P_n(\cos\theta)$$

Coefficients of Legendre polynomials of given n must be equal for this equation to hold, so

$$n = 1 \qquad \frac{p}{4\pi\varepsilon} \frac{1}{R^2} + A_1 R = \frac{1}{R^2} B_1 \qquad (1)$$

$$n \neq 1 \qquad A_n R^n = \frac{B_n}{R^{n+1}}$$

$$\Leftrightarrow \qquad A_n = \frac{B_n}{R^{2n+1}} \qquad (2)$$

Also, $\quad \varepsilon_r \varepsilon_0 \left.\dfrac{\partial \phi_{in}}{\partial r}\right|_{r=R} = \varepsilon_0 \left.\dfrac{\partial \phi_{out}}{\partial r}\right|_{r=R}$

$$\Rightarrow \quad \Big[-\frac{2\rho\varepsilon_r}{4\pi\varepsilon_0\varepsilon_r}\frac{1}{R^3} + \varepsilon_r \sum_n A_n\, n\, R^{n-1} = \sum (-1)(n+1)\frac{B_n}{R^{n+2}}$$

$n=1$
$$-\frac{2\rho}{4\pi\varepsilon_0}\frac{1}{R^3} + \varepsilon_r A_1 = -\frac{2B_1}{R^3} \qquad (3)$$

$n \neq 1$
$$\varepsilon_r\, n\, A_n R^{n-1} = -(n+1)\frac{B_n}{R^{n+2}}$$

$$\Leftrightarrow \qquad A_n = -\frac{n+1}{\varepsilon_r n}\frac{B_n}{R^{2n+2}} \qquad (4)$$

(2) and (4) can not be satisfied for all n and ε_r unless

$$A_n = B_n = 0 \quad \text{for } n \neq 1.$$

Now solve (1) and (3) for A_1 and B_1:

$$\left.\begin{aligned} \frac{\rho}{4\pi\varepsilon_0}\frac{1}{R^3} + \varepsilon_r A_1 &= \frac{\varepsilon_r B_1}{R^3} \\ -\frac{2\rho}{4\pi\varepsilon_0}\frac{1}{R^3} + \varepsilon_r A_1 &= -\frac{2B_1}{R^3} \end{aligned}\right\rangle -$$

$$\Rightarrow \quad (\varepsilon_r+2)B_1 = \frac{3\rho}{4\pi\varepsilon_0} \qquad \Leftrightarrow \qquad B_1 = \frac{3\rho}{4\pi\varepsilon_0}\left(\frac{1}{\varepsilon_r+2}\right)$$

and $\quad A_1 = \dfrac{B_1}{R^3} - \dfrac{\rho}{4\pi\varepsilon_0\varepsilon_r}\dfrac{1}{R^3}$

$$= \frac{\rho}{4\pi\varepsilon_0 R^3}\left(\frac{3}{\varepsilon_r+2} - \frac{1}{\varepsilon_r}\right) \quad \Leftrightarrow \quad A_1 = \frac{\rho}{4\pi\varepsilon_0 R^3}\left(\frac{2\varepsilon_r - 2}{\varepsilon_r(\varepsilon_r+2)}\right)$$

$$\Rightarrow \quad \Phi_{in} = \frac{p\cos\theta}{4\pi\varepsilon\ r^2} + \frac{p\cos\theta}{4\pi\varepsilon_0\varepsilon_r R^3}\ r\ \frac{2\varepsilon_r - 2}{\varepsilon_r + 2}$$

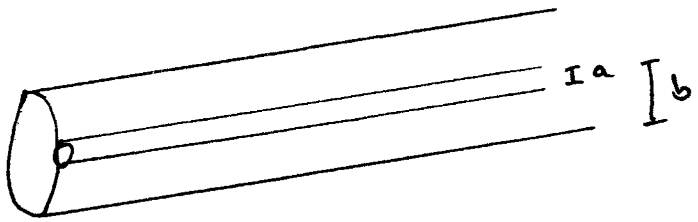
$$= \frac{p\cos\theta}{4\pi\varepsilon\ r^2}\left[1 + \frac{r^3}{R^3}\left(\frac{2\varepsilon_r - 2}{\varepsilon_r + 2}\right)\right]$$

and

$$\Phi_{out} = \frac{3p\cos\theta}{4\pi\varepsilon_0\ r^2}\left(\frac{1}{\varepsilon_r + 2}\right)$$

$\varepsilon_r = 1$ (no sphere) gives $\Phi_{in} = \Phi_{out} = \dfrac{p\cos\theta}{4\pi\varepsilon_0 r^2}$ (✓)

Problem 2 SOLUTION



magnetic fields using Ampère's law

$r > b \qquad \vec{B} = 0$

$a < r < b \qquad \oint \vec{B}\, d\vec{l} = \mu_0 I \Rightarrow 2\pi s B = \mu_0 I$

$$\Rightarrow \vec{B} = \frac{\mu_0 I}{2\pi s}\hat{\phi}$$

$r < a$

$$2\pi s B = \mu_0 I \frac{s^2}{a^2} \Rightarrow \vec{B} = \frac{\mu_0}{2\pi}\frac{I s}{a^2}\hat{\phi}$$

$$W = \frac{1}{2\mu_0}\iiint B^2\, dV$$, so per unit length

$$\frac{W}{l} = \frac{\mu_0}{2}\frac{I^2}{(2\pi)^2}2\pi\left\{\int_0^a \frac{s^2}{a^4} s\, ds + \int_a^b \frac{1}{s^2} s\, ds\right\}$$

$$= \frac{\mu_0}{4\pi} I^2 \left\{\left[\frac{1}{4}\frac{s^4}{a^4}\right]_0^a + \left[\ln s\right]_a^b\right\}$$

$$= \frac{\mu_0}{16\pi} I^2 + \frac{\mu_0}{4\pi}\ln\frac{b}{a} I^2$$

Now use $W = \frac{1}{2} L I^2$, so $\frac{L}{l} = \frac{\mu_0}{8\pi} + \frac{\mu_0}{2\pi}\ln b/a$

Vanishes for cylindrical shell

Delivered-To: lalla@phys.columbia.edu
X-Authentication-Warning: pegasus.phys.columbia.edu: westerhoff owned process doing -bs
Date: Tue, 13 Jan 2004 11:30:22 -0500 (EST)
From: Stefan Westerhoff <westerhoff@nevis.columbia.edu>
X-X-Sender: westerhoff@pegasus.phys.columbia.edu
To: Brian Cole <cole@nevis.columbia.edu>
Cc: "Lalla R. Grimes" <lalla@phys.columbia.edu>
Subject: qual grading

Hi Brian,

there is a tricky problem with question 2 on the
E&M part of the quals.
The self-inductance of the configuration of question 2
as given in the problem has a typo - the
contribution of the solid cylinder is mu_0/8pi,
not mu_0/4pi (the problem I submitted has the correct
inductance, but I was at a collaboration meeting in
Utah when Lalla's request for proof-reading reached me,
so I had no chance to catch this).

It would not be a big deal for most problems, but
for this one, it unfortunately is.
If you calculate the self-inductance L using the energy
of the magnetic field and then using
W = 1/2 L I^2
you get the correct answer (mu_0/8pi). However, if
you calculate the magnetic flux Phi and then use
Phi = L I
to get the self-inductance L, you get an incorrect
answer (the formula does not apply since the current
is not confined to a single path, at least not in a
trivial way -- one would have to split the finite wire
into lots of small wires etc etc ...).
This incorrect answer is unfortunately (how much more
Murphy can there be ?) the answer given in the problem...
Since lots of students used the wrong way to get to
the answer, they did not wonder why they were off
by a factor of 2 (as they should have -- that was my
reason to actually give the solution...).

Bottom line, this is somewhat tricky to grade.
I guess I cannot really subtract any points for the
wrong solution, can't I ? Maybe just a remark that
this is not the way to calculate it ?

Stefan

--

Section 2, Question 3

A. Mueller

5

(2) E+M

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Two halves of a spherical metallic shell of radius R and infinite conductivity are separated by a very small insulating gap. An alternating potential is applied between the two halves of the sphere so that the potentials are $\pm V\cos\omega t$. In the long wavelength limit find the angular distribution of radiated power and the total radiated power from the sphere.

Solution:

This is a dipole problem, because of the long wavelength limit so

$$\frac{dP(t)}{d\Omega} = \frac{1}{4\pi c^3}\left|\ddot{\vec{p}}(t-r/c)\times\vec{n}\right|^2$$

where $\vec{p}$ is the dipole moment and $\vec{n}=\vec{r}/r$.

Pretend for the moment that we are dealing with a static two-hemisphere problem. Then outside the sphere

$$\phi(\vec{r}) = \sum_{\ell=0}^{\infty}\phi_\ell\, P_\ell(\cos\theta)\frac{1}{r^{\ell+1}}.$$

We are interested in ϕ_1 which is determined by

$$\underbrace{\int_{-1}^{1} d\cos\theta\, P_1(\cos\theta)\, V\,\epsilon(\cos\theta)}_{V} = \frac{1}{R^2}\phi_1 \underbrace{\int_{-1}^{1} P_1^2(\cos\theta)\, d\cos\theta}_{2/3}$$

$$\Rightarrow \quad \phi_1 = \frac{3}{2}VR^2$$

So $\phi = \frac{3}{2}V\frac{R^2}{r^2}\cos\theta$ outside sphere

But $\phi = \frac{\vec{p}\cdot\vec{n}}{r^2} \Rightarrow \boxed{\vec{p} = \frac{3}{2}VR^2\vec{e}_3}$

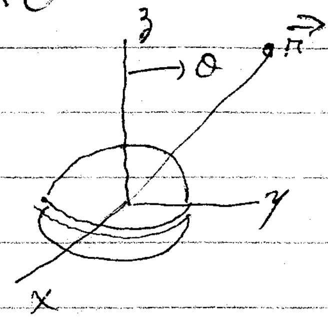
A. Mueller

Now go back to time-dependent problem

$$\vec{p} = \frac{3}{2} V R^2 \vec{e}_z \cos\omega t$$

$$\ddot{\vec{p}} = -\frac{3}{2} V R^2 \omega^2 \vec{e}_z \cos\omega t$$

$$\frac{dP(t)}{d\Omega} = \frac{9R^4V^2\omega^4}{16\pi c^3}\cos^2[\omega(t-r/c)]\sin^2\theta$$



$$P = \frac{9R^4V^2\omega^4}{16\pi c^3}\cos^2\omega(t-r/c)\underbrace{\int_{-1}^{1}\sin^2\theta\, d\cos\theta}_{4/3}\underbrace{\int_0^{2\pi} d\varphi}_{2\pi}$$

$$\boxed{P = \frac{3R^4V^2\omega^4}{2c^3}\cos^2\omega(t-r/c)}$$

Section 2, Question 4

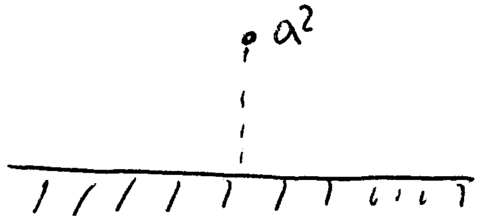
1. Solution:

Alemel

Introduce $z = x + iy$;

By conformal transformation $u = z^2$;

the system is mapped onto



charge

The potential $\varphi(u) = -2q \, \mathrm{Re} \ln\left(\dfrac{u - ia^2}{u + ia^2}\right)$;

image charge;

Thus; $\varphi(z) = -2q \, \mathrm{Re} \ln\left(\dfrac{z^2 - ia^2}{z^2 + ia^2}\right)$;

Electric fields:

$$E_x = -\frac{\partial \varphi}{\partial x} = \cancel{-2q} - \mathrm{Re}\frac{\partial \varphi}{\partial z} = 4q \, \mathrm{Re}\left(\frac{z}{z^2 - ia^2} - \frac{z}{z^2 + ia^2}\right);$$

$$E_y = -\frac{\partial \varphi}{\partial y} = 4q \, \mathrm{Im}\left(\frac{z}{z^2 - ia^2} - \frac{z}{z^2 + ia^2}\right);$$

Surface charge on the horisontal segment: $y = 0$:

$$4\pi \sigma(x) = E_y = 4q \, \mathrm{Im}\left(\frac{x}{x^2 - ia^2} - \frac{x}{x^2 + ia^2}\right) = 8q \frac{x a^2}{x^4 + a^4};$$

$$\sigma(x) = \frac{2q}{\pi} \frac{x a^2}{x^4 + a^4}; \qquad x > 0;$$

Analogously (or by symmetry):

$$\sigma(y) = \frac{2q}{\pi} \frac{y a^2}{y^4 + a^4}; \qquad y > 0;$$

DIRAC

Electromagnetic Waves – Solution   Section 2, Question #5

(a) $\frac{\partial\rho}{\partial t} + \operatorname{div}\vec{J} = 0$ where $\rho = \frac{1}{4\pi}\left[\operatorname{div}\vec{E} + \mu^2\Phi\right]$

& $\vec{J} = \frac{c}{4\pi}\left[\operatorname{curl}\vec{B} + \mu^2\vec{A} - \frac{1}{c}\frac{\partial\vec{E}}{\partial t}\right]$.

$$\therefore \frac{1}{4\pi}\left[\operatorname{div}\frac{\partial\vec{E}}{\partial t} + \mu^2\frac{\partial\Phi}{\partial t}\right] + \frac{c}{4\pi}\left[\underbrace{\operatorname{div}\operatorname{curl}\vec{B}}_{0} + \mu^2\operatorname{div}\vec{A} - \frac{1}{c}\operatorname{div}\frac{\partial\vec{E}}{\partial t}\right] = 0.$$

cancel

$$\therefore \frac{\mu^2}{4\pi}\left\{\frac{\partial\Phi}{\partial t} + c\operatorname{div}\vec{A}\right\} = 0. \qquad \therefore \boxed{\frac{1}{c}\frac{\partial\Phi}{\partial t} + \operatorname{div}\vec{A} = 0}.$$

(b) $\operatorname{div}\vec{E} + \mu^2\Phi = 4\pi\rho \Rightarrow \operatorname{div}\left(-\nabla\Phi - \frac{1}{c}\frac{\partial\vec{A}}{\partial t}\right) + \mu^2\Phi = 4\pi\rho$.

$$\therefore -\nabla^2\Phi - \frac{1}{c}\frac{\partial}{\partial t}\underbrace{\operatorname{div}\vec{A}}_{-\frac{1}{c}\frac{\partial\Phi}{\partial t}} + \mu^2\Phi = 4\pi\rho \Rightarrow \boxed{+\frac{1}{c^2}\frac{\partial^2\Phi}{\partial t^2} - \nabla^2\Phi + \mu^2\Phi = 4\pi\rho}$$

$$\operatorname{curl}\vec{B} + \mu^2\vec{A} = \frac{4\pi}{c}\vec{J} + \frac{1}{c}\frac{\partial\vec{E}}{\partial t} \Rightarrow \underbrace{\operatorname{curl}\operatorname{curl}\vec{A}}_{(\operatorname{grad}\underbrace{\operatorname{div}\vec{A}}_{-\frac{1}{c}\frac{\partial\Phi}{\partial t}} - \nabla^2\vec{A})} + \mu^2\vec{A} = \frac{4\pi}{c}\vec{J} + \frac{1}{c}\frac{\partial}{\partial t}\left(-\nabla\Phi - \frac{1}{c}\frac{\partial\vec{A}}{\partial t}\right)$$

cancel

$$\therefore \boxed{+\frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2} - \nabla^2\vec{A} + \mu^2\vec{A} = \frac{4\pi}{c}\vec{J}}.$$

(c) $\rho = 0$ and $\vec{J} = 0 \Rightarrow \left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2 + \mu^2\right)\left[\begin{Bmatrix}\vec{A}_0\\ \Phi_0\end{Bmatrix} e^{i(kz-\omega t)}\right] = 0$,

$$\therefore \left(\frac{-\omega^2}{c^2} + k^2 + \mu^2\right) = 0. \qquad \therefore \boxed{k = \sqrt{\frac{\omega^2}{c^2} - \mu^2}}.$$

$\boxed{\omega_{cutoff} = \mu c}$. $\omega > \omega_{cutoff} \Rightarrow k$ pure real (pure propagation)

$\omega < \omega_{cut\text{-}off} \Rightarrow k$ pure imaginary (pure attenuation)

$$v_{phase} = \frac{\omega}{k} = \frac{\omega}{\sqrt{\frac{\omega^2}{c^2} - \mu^2}} \Rightarrow \boxed{v_{phase} = c\frac{\omega}{\sqrt{\omega^2 - \mu^2c^2}}}.$$

(d) $\vec{A}_0 = A_0\hat{z}$.

$\frac{1}{c}\frac{\partial\Phi}{\partial t} + \operatorname{div}\vec{A} = 0 \Rightarrow \frac{-i\omega}{c}\Phi_0 + ikA_0 = 0 \Rightarrow \boxed{\Phi_0 = \frac{kc}{\omega}A_0}$ (k given above)

$\begin{Bmatrix}\vec{E}(z,t)\\ \vec{B}(z,t)\end{Bmatrix} = \begin{Bmatrix}\vec{E}_0\\ \vec{B}_0\end{Bmatrix} e^{i(kz-\omega t)}$ — Take real part.

$\vec{B} = \operatorname{curl}\vec{A} \Rightarrow \vec{B}_0 = ik\hat{z}\times(A_0\hat{z}) = 0 \Rightarrow \boxed{\vec{B}_0 = 0}$.

$\vec{E} = -\nabla\Phi - \frac{1}{c}\frac{\partial\vec{A}}{\partial t} \Rightarrow \vec{E}_0 = -ik\hat{z}\Phi_0 + \frac{i\omega}{c}A_0\hat{z}$

$$\therefore \vec{E}_0 = i\hat{z}\left[\frac{\omega}{c}A_0 - k\left(\frac{kc}{\omega}A_0\right)\right] = i\hat{z}A_0\frac{\omega}{c}\left[1 - \frac{k^2c^2}{\omega^2}\right] \Rightarrow \boxed{\vec{E}_0 = i\hat{z}A_0\mu^2\frac{c}{\omega}}$$