

Columbia University
Department of Physics
QUALIFYING EXAMINATION

Monday, January 13, 2014

3:10PM to 5:10PM

Classical Physics

Section 2. Electricity, Magnetism & Electrodynamics

Two hours are permitted for the completion of this section of the examination. Choose 4 problems out of the 5 included in this section. (You will not earn extra credit by doing an additional problem). Apportion your time carefully.

Use separate answer booklet(s) for each question. Clearly mark on the answer booklet(s) which question you are answering (e.g., Section 2 (Electricity etc.), Question 2, etc.).

Do **NOT** write your name on your answer booklets. Instead, clearly indicate your **Exam Letter Code**.

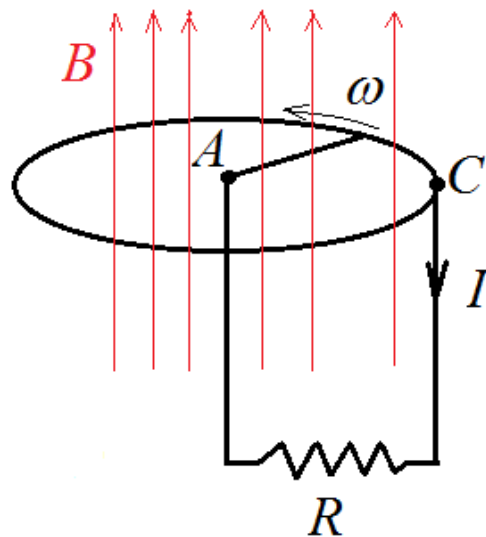
You may refer to the single handwritten note sheet on $8\frac{1}{2}$ " \times 11" paper (double-sided) you have prepared on Classical Physics. The note sheet cannot leave the exam room once the exam has begun. This note sheet must be handed in at the end of today's exam. Please include your Exam Letter Code on your note sheet. No other extraneous papers or books are permitted.

Simple calculators are permitted. However, the use of calculators for storing and/or recovering formulae or constants is NOT permitted.

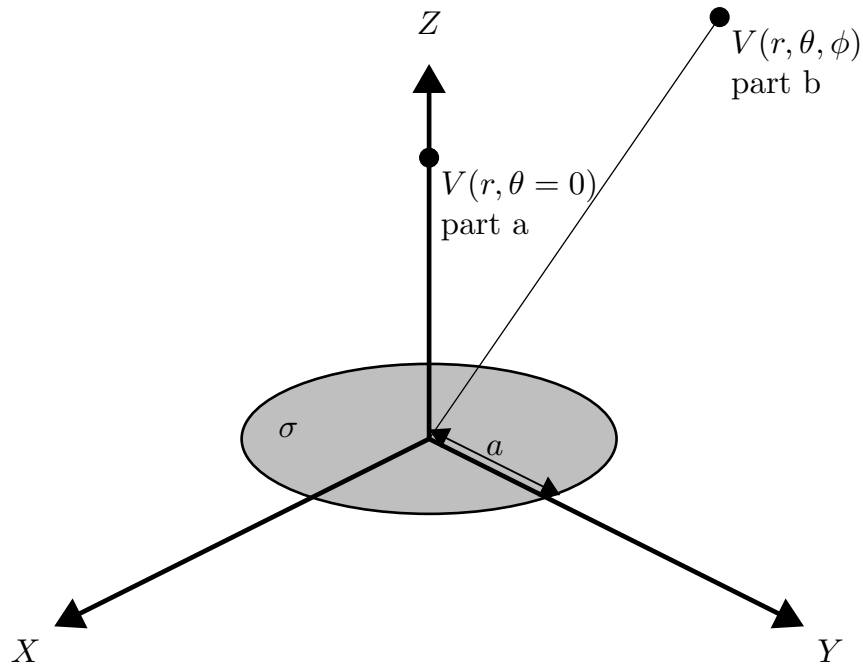
Questions should be directed to the proctor.

Good Luck!

1. One end of a conducting rod rotates with angular velocity ω in a circle of radius r making contact with a horizontal, conducting ring of the same radius. The other end of the rod is fixed. Stationary conducting wires connect the fixed end of the rod (A) and a fixed point on the ring (C) to either end of a resistance R . A uniform vertical magnetic field \vec{B} passes through the ring.
- (a) Find the current I flowing through the resistor.
 - (b) What is sign of the current if positive I corresponds to flow in the direction of the arrow in the figure?
 - (c) What torque must be applied to the rod to maintain its rotation at the constant angular rate ω ?



2. Consider a very thin rubber sheet of radius a containing a uniform surface charge density σ .
- (a) Calculate an exact expression for the electric potential $V(r, \theta = 0)$ at an arbitrary position r above the center of the disk.
 - (b) Calculate the electric potential $V(r, \theta, \phi)$ at all points in space for $r \gtrsim a$



3. A rectangular copper strip 1.5cm wide and 0.10cm thick carries a current of 5.0 A. A 1.2 T magnetic field is applied perpendicular to the strip. Find the resulting Hall voltage. The molar mass of copper is 63.5 g, and the density of copper is 8.95 g/cm³. Assume each copper atom contributes one free electron to the body of the material.

4. Consider an infinitesimally thin charged disk of radius R and uniform surface charge density σ that is in the xy plane centered on the origin.
- (a) Calculate the potential along the z axis from Coulomb's law.
 - (b) Calculate the monopole moment, dipole moment and quadrupole moment of the disk. These are defined as

$$Q = \int d^3r \rho(r)$$
$$x\vec{P} = \int d^3r \rho(r) \vec{r}$$
$$Q_{ij} = \int d^3r \rho(r) [3r_i r_j - r^2 \delta_{ij}]$$

5. Consider a conductor in which the free current density and the electric field are related by the linear relationship

$$\vec{j}_{free} = \vec{g} \times \vec{E} = g(\hat{z} \times \vec{E})$$

where \vec{g} is a given real constant vector directed along the z-axis ($\vec{g} = g\hat{z}$ with $g > 0$). The conductor's polarization and magnetization vectors are both zero ($\vec{P} = 0, \vec{M} = 0$).

Consider a monochromatic plane wave traveling through such a conductor in the +z direction (propagation parallel to the vector \vec{g}):

$$\begin{bmatrix} \vec{E}(z, t) \\ \vec{B}(z, t) \end{bmatrix} = \text{Real Part} \left\{ \begin{bmatrix} \vec{E}_0 \\ \vec{B}_0 \end{bmatrix} e^{i(kz - \omega t)} \right\}$$

where ω is the given (real) angular frequency of the wave. \vec{E}_0, \vec{B}_0 and k are constants to be determined.

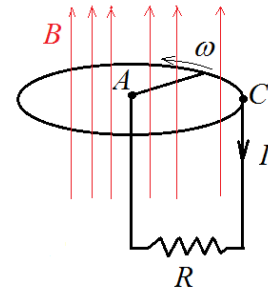
- Find the possible values for the index of refraction ($N = kc/\omega$) for this wave. Express your answers in terms of g and ω .
- Show that there is a cut-off frequency ω_c such that for $\omega > \omega_c$ all the waves which have definite N propagate without attenuation. Also show, for $\omega < \omega_c$, that only one of these waves propagates and that the other wave does not propagate but just undergoes attenuation. Find the value for ω_c , and briefly explain your reasoning.
- For each possible value of N , find the corresponding electric-field polarization \vec{E}_0 .

N. Christ

November 30, 2013

Quals E&M Problem

1. One end of a conducting rod rotates with angular velocity ω in a circle of radius r making contact with a horizontal, conducting ring of the same radius. The other end of the rod is fixed. Stationary conducting wires connect the fixed end of the rod (A) and a fixed point on the ring (C) to either end of a resistance R . A uniform vertical magnetic field \vec{B} passes through the ring.



- (a) Find the current I flowing through the resistor. [10 points]
- (b) What is sign of the current if positive I corresponds to flow in the direction of the arrow in the figure? [2 points]
- (c) What torque must be applied to the rod to maintain its rotation at the constant angular rate ω ? [8 points]

Suggested Solution

- (a) Apply the integrated version of Faraday's law to the closed loop \mathcal{L} made up of the segment of the circle connecting the moving end of the rod and the point C, moving counter clockwise (when viewed from above) from that end to C, the segment from C to A containing the resistor R and then the rod itself:

$$\oint_{\mathcal{L}} d\vec{r} \cdot \vec{E} = -\frac{1}{c} \frac{d}{dt} \Phi_B$$

where $\Phi_B(t)$ is the flux of magnetic field passing through that loop. Evaluating the left and right hand sides of this equation:

$$IR = \frac{1}{c} \frac{\omega r^2}{2} B \quad \text{or} \quad I = \frac{r^2 B \omega}{2cR}$$

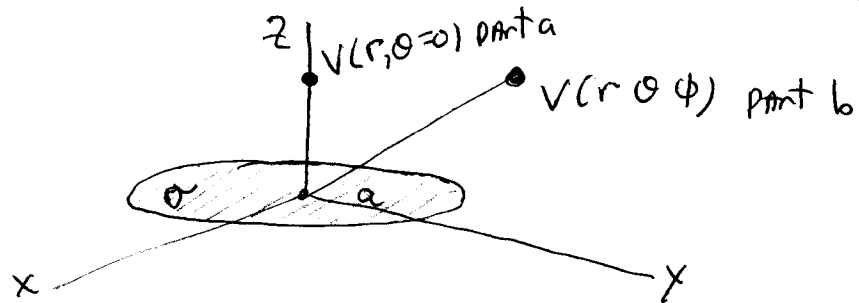
- (b) As found in (a), I is positive.
- (c) Consider a segment of the rod of length dx a distance x from the pivot. Because of the flowing current this segment will contain a charge dQ moving at velocity v where $I = dQ/(dx/v) = vdQ/dx$. The torque τ exerted by the magnetic field on the rod carrying the current I is then given by:

$$\tau = \int_0^r x \frac{v}{c} B \frac{dQ}{dx} dx = \frac{IB}{c} \int_0^r x dx = \frac{IBr^2}{2c} = \frac{r^4 B^2 \omega}{4c^2 R}$$

pointed downward. Thus, $-\vec{\tau}$ must be applied to the rod to maintain its rotational motion.

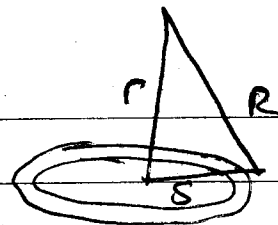
Hailey E-M

Consider a very thin rubber sheet of radius a containing a uniform surface charge density σ .



- Calculate an exact expression for the electric potential $V(r, \theta=0)$ at an arbitrary position r above the center of the disk.
- Calculate the electric potential $V(r, \theta, \phi)$ at all points in space for $r \gg a$.

Solution E-M Hailey



$$a.) V = \int \frac{dq}{R} = \int_0^a \frac{2\pi s ds \sigma}{\sqrt{r^2 + s^2}}$$

$$b.) V = 2\pi\sigma(\sqrt{r^2 + a^2} - r) \quad \text{Ans } \frac{2}{2}$$

b.) with the potential $V=0$ at $r=\infty$

$$V(r, \theta) = \sum b_n r^{-(n+1)} P_n(\cos\theta)$$

for Azimuthal symmetry

For $r \geq a$ as indicated in problem we can match the b_n by using expression from part a i.e

$$V(r, \theta) = 2\pi\sigma \left\{ \frac{a^2}{2r} - \frac{1}{8} \frac{a^4}{r^3} + \frac{1}{16} \frac{a^6}{r^5} - \dots \right\}$$

By inspection the 1st few terms are

$$\text{Ans } \frac{2}{2} V(r, \theta) = 2\pi\sigma \left\{ \frac{a^2}{2} r^{-1} - \frac{1}{8} a^4 r^{-3} P_2(\cos\theta) + \frac{1}{16} a^6 r^{-5} P_4(\cos\theta) - \dots \right\}$$

In general:

$$V(r, \theta) = 2\pi\sigma \sum_{\substack{\text{even} \\ n}} \frac{a^{n+2}}{r^{n+1}} \frac{q \dots (q - n/2)}{(n/2 + 1)!} P_n(\cos\theta)$$

$$(q = \frac{1}{2})$$

Quals Problem #2

January 2014

Problem

A rectangular copper strip 1.5 cm wide and 0.10 cm thick carries a current of 5.0 A. A 1.2 T magnetic field is applied perpendicular to the strip. Find the resulting Hall voltage. The molar mass of copper is 63.5 g, and the density of copper is 8.95 g/cm³. Assume each copper atom contributes one free electron to the body of the material.

Suggested Solution to Quas Problem #2

January 2014

Solution

One mole of copper has the following volume:

$$V = \frac{m}{\rho} = \frac{63.5 \text{ g}}{8.95 \text{ g/cm}^3} = 7.09 \text{ cm}^3 \quad (1)$$

Therefore, the number density of free electrons in copper is:

$$n = \frac{N_A}{V} = \frac{6.02 \times 10^{23} \text{ electrons}}{7.09 \text{ cm}^3} \quad (2)$$

which can be straightforwardly converted to 8.48×10^{28} electrons/m³. The Hall voltage is:

$$V_H = \frac{IB}{nqt} = \frac{(5.0 \text{ A})(1.2 \text{ T})}{(8.48 \times 10^{28} \text{ m}^{-3})(1.60 \times 10^{-19} \text{ C})(0.10 \times 10^{-2} \text{ m})} = 0.44 \text{ } \mu\text{V} \quad (3)$$

Electromagnetism Qualls Problem

Fall 2013

Robert Mawhinney

Consider an infinitesimally thin charged disk of radius R and uniform surface charge density σ that is in the xy plane centered on the origin.

1. Calculate the potential along the z axis from Coulomb's law.
2. Calculate the monopole moment, dipole moment and quadrapole moment of the disk. These are defined as

$$Q = \int d^3r \rho(r) \quad (1)$$

$$\vec{P} = \int d^3r \rho(r) \vec{r} \quad (2)$$

$$Q_{ij} = \int d^3r \rho(r) [3r_i r_j - r^2 \delta_{ij}] \quad (3)$$

Solution

1.

$$\phi(x) = \int d^2x \frac{\sigma}{\sqrt{z^2 + x^2 + y^2}} \quad (4)$$

$$= \sigma \int_0^R 2\pi\rho d\rho \frac{1}{\sqrt{z^2 + \rho^2}} \quad (5)$$

$$= 2\pi\sigma \sqrt{z^2 + \rho^2} \Big|_0^R \quad (6)$$

$$= 2\pi\sigma \left(\sqrt{z^2 + R^2} - |z| \right) \quad (7)$$

2. The monopole moment is just the total charge, which is $\sigma\pi R^2$. The dipole moment vanishes by symmetry. The quadrupole moment is zero, unless $i = j$. We need

$$\int d^2x \sigma x^2 = \int d^2x \sigma y^2 \quad (8)$$

by symmetry. Thus we have

$$\int d^2x \sigma x^2 = \frac{1}{2} \int d^2x \sigma (x^2 + y^2) \quad (9)$$

$$= \pi\sigma \int_0^R d\rho \rho^3 \quad (10)$$

$$= \frac{\pi\sigma}{4} \rho^4 \Big|_0^R \quad (11)$$

$$= \frac{\pi\sigma}{4} R^4 \quad (12)$$

$$(13)$$

This gives

$$Q_{ij} = \frac{\pi\sigma R^4}{4} \begin{pmatrix} 3-2 & 0 & 0 \\ 0 & 3-2 & 0 \\ 0 & 0 & 0-2 \end{pmatrix} = \frac{\pi\sigma R^4}{4} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} \quad (14)$$

E&M Qualifying Examination - 2014
Allan Blaer

AN ANISOTROPIC CONDUCTOR

Consider a conductor in which the free current density and the electric field are related by the linear relationship

$$\vec{J}_{free} = \vec{g} \times \vec{E} = g(\hat{z} \times \vec{E})$$

where \vec{g} is a given real constant vector directed along the z-axis ($\vec{g} = g\hat{z}$, with $g > 0$). The conductor's polarization and magnetization vectors are both zero ($\vec{P} = 0$, $\vec{M} = 0$).

Consider a monochromatic plane wave traveling through such a conductor in the +z direction (propagation parallel to the vector \vec{g}):

$$\begin{bmatrix} \vec{E}(z, t) \\ \vec{B}(z, t) \end{bmatrix} = \text{Real Part} \left\{ \begin{bmatrix} \vec{E}_0 \\ \vec{B}_0 \end{bmatrix} e^{i(kz - \omega t)} \right\}$$

where ω is the given (real) angular frequency of the wave. \vec{E}_0 , \vec{B}_0 , and k are constants to be determined.

- Find the possible values for the index of refraction ($N = kc/\omega$) for this wave. Express your answers in terms of g and ω .
- Show that there is a cut-off frequency ω_c such that for $\omega > \omega_c$ all the waves which have definite N propagate without attenuation. Also show, for $\omega < \omega_c$, that only one of these waves propagates and that the other wave does not propagate but just undergoes attenuation. Find the value for ω_c , and briefly explain your reasoning.
- For each possible value of N , find the corresponding electric-field polarization \vec{E}_0 .

E & M Anisotropic Conductor.
 Solution - Allan Blaes.

(1)

$$(a) \text{curl } \vec{E} = -\frac{1}{c} \frac{\partial \vec{B}}{\partial t} \implies i k \hat{z} \times \vec{E} = +\frac{i\omega}{c} \vec{B}$$

$$\therefore \vec{B} = \frac{bc}{\omega} (\hat{z} \times \vec{E})$$

$$N \equiv \frac{bc}{\omega} \implies \vec{B} = N (\hat{z} \times \vec{E})$$

$$\text{curl } \vec{H} = \frac{4\pi}{c} \vec{J}_{\text{free}} + \frac{1}{c} \frac{\partial \vec{D}}{\partial t}$$

$$\vec{B} = \vec{H}, \vec{D} = \vec{E}, \text{ and } \vec{J}_{\text{free}} = g (\hat{z} \times \vec{E})$$

$$\therefore \text{curl } \vec{B} = \frac{4\pi}{c} g (\hat{z} \times \vec{E}) + \frac{1}{c} \frac{\partial \vec{E}}{\partial t}$$

$$\therefore i k \hat{z} \times \vec{B} = \frac{4\pi}{c} g (\hat{z} \times \vec{E}) - \frac{i\omega}{c} \vec{E}$$

$$\{ i k \hat{z} \times [N \hat{z} \times \vec{E}] \}$$

← Multiply by $(\frac{-ic}{\omega})$.

$$N^2 \hat{z} \times (\hat{z} \times \vec{E}) = -i \frac{4\pi}{\omega} g (\hat{z} \times \vec{E}) - \vec{E}$$

$$[\hat{z}(\vec{E} \cdot \hat{z}) - \vec{E}(\hat{z} \cdot \hat{z})]$$

$$[\hat{z} E_z - \vec{E}]$$

$$\therefore (N^2 - 1) \vec{E} - N^2 \hat{z} E_z - i \frac{4\pi}{\omega} g (\hat{z} \times \vec{E}) = 0 \implies \begin{pmatrix} N^2 - 1 & +i \frac{4\pi g}{\omega} & 0 \\ -i \frac{4\pi g}{\omega} & N^2 - 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix} = 0$$

(matrix \mathcal{M})

For a non-trivial solution to exist, $\det \mathcal{M} = 0$

$$\therefore \det \mathcal{M} = (-1) \left[(N^2 - 1)^2 - \left(\frac{4\pi g}{\omega} \right)^2 \right] = 0 \implies N^2 - 1 = \pm \frac{4\pi g}{\omega}$$

$$\therefore N_{\pm}^2 = 1 \pm \frac{4\pi g}{\omega}$$

(continued)

(2)

(Solution continued)

$$(b) N_+^2 = 1 + \frac{4\pi g}{\omega} > 0 \text{ for all } \omega. \left. \begin{array}{l} N_+ \text{ is pure real for all } \omega \\ \text{and one has propagation with} \\ \text{no attenuation.} \end{array} \right\}$$

$$N_-^2 = 1 - \frac{4\pi g}{\omega} \left\{ \begin{array}{l} > 0 \text{ for } \omega > 4\pi g \Rightarrow N_- \text{ pure real} \\ < 0 \text{ for } \omega < 4\pi g \Rightarrow N_- \text{ pure imaginary} \end{array} \right\}$$

$$\therefore \omega_{\text{cutoff}} = 4\pi g$$

$$\text{with } \omega > \omega_{\text{cutoff}} \Rightarrow N_- \text{ pure real (propagation with no attenuation)}$$

$$\text{and with } \omega < \omega_{\text{cutoff}} \Rightarrow N_- \text{ pure imaginary (just attenuation and no propagation)}$$

$$(c) N_{\pm}^2 = 1 \pm \frac{4\pi g}{\omega} \quad \therefore M\vec{E} = \begin{pmatrix} \pm \frac{4\pi g}{\omega} & \pm i \frac{4\pi g}{\omega} & 0 \\ -i \frac{4\pi g}{\omega} & \pm \frac{4\pi g}{\omega} & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix} = 0$$

$$\therefore E_z = 0 \text{ and } \underbrace{\pm E_x + i E_y = 0}_{E_y = \pm i E_x}$$

$$\therefore \vec{E}_0 \sim (\hat{x} \pm i\hat{y}) \text{ for } N_{\pm} \text{ (2 circular polarizations)} \\ \text{(transverse to } \hat{z})$$