Columbia University
Department of Physics
QUALIFYING EXAMINATION

Wednesday, January 11, 2017
3:00PM to 5:00PM
Modern Physics
Section 4. Relativity and Applied Quantum Mechanics

Two hours are permitted for the completion of this section of the examination. Choose 4 problems out of the 5 included in this section. (You will not earn extra credit by doing an additional problem). Apportion your time carefully.

Use separate answer booklet(s) for each question. Clearly mark on the answer booklet(s) which question you are answering (e.g., Section 4 (Relativity and Applied QM), Question 2, etc.).

Do **NOT** write your name on your answer booklets. Instead, clearly indicate your **Exam Letter Code**.

You may refer to the single handwritten note sheet on $8\frac{1}{2}$" $\times$ 11" paper (double-sided) you have prepared on Modern Physics. The note sheet cannot leave the exam room once the exam has begun. This note sheet must be handed in at the end of today's exam. Please include your Exam Letter Code on your note sheet. No other extraneous papers or books are permitted.

Simple calculators are permitted. However, the use of calculators for storing and/or recovering formulae or constants is NOT permitted.

Questions should be directed to the proctor.

Good Luck!

1. A hoop of radius R is floating in space. In an inertial lab frame, the hoop is observed to rotate about its center with angular velocity $\vec{\Omega}$ perpendicular to the plane of the hoop. Rabbit Carlo of mass m is comfortably sitting on the inner side of the hoop. His sitting is helped by the effective "gravity" due to centrifugal acceleration (like in the rotating spaceship in Stanley Kubrick's "2001: A Space Odyssey"). In classical Newtonian mechanics, Carlo would find that his weight is mv^2/R where $v = \Omega R \ll c$. Obtain the relativistic generalization of this formula that is valid for any v.

2. If the motion of an electron (of charge $-e$) around a nucleus (of charge $+e$ and mass $m \gg m_e$) is confined to a plane by certain constraints, we call such system a "two-dimensional (2D) hydrogen atom". This problem examines some properties of the 2D hydrogen atom in polar coordinates (r, ϕ). The potential for the system is

$$V(r) = -\frac{e^2}{4\pi\epsilon_0 r}.$$

   (a) In analogy with the three dimensional hydrogen atom, let us propose a trial wavefunction for the ground state in the form

$$\psi_g(r, \phi) = Ae^{-\alpha r}, \tag{1}$$

   where α is a parameter. Using the variation principle with the wavefunction of this form, evaluate the ground state energy of the 2D hydrogen atom. Is it smaller or larger than the ground state energy of the 3D hydrogen atom?

   (b) The above trial wavefunction (Equation 1) is indeed the correct ground state wavefunction of the 2D hydrogen atom. The ground state corresponds to quantum numbers $n = 1$, $l = 0$. The wavefunction of the excited state with $n = 2$, $l = 0$ has the form,

$$\psi_{2,0}(r, \phi) = B(1 - \beta r)e^{-\gamma r},$$

   where B, β, and γ are some constants. Express β in terms of α and γ. [Note: you are not being asked to evaluated B or γ, or use the calculated value of α. You are just being asked to obtain a simple relationship between α, β, and γ.]

3. An entrepreneur decides to operate a spacewash. Although the device is only 100 meters long, she advertises her spacewash as "the only one that simultaneously washes front and back of a WizzFlizz 200^{TM}." According to the catalog, the WizzFlizz 200^{TM} is 200 meters long.

   (a) An inspector arrived at spacewash to check how it operates. What is the minimum speed v_0 (in units of c) at which the WizzFlizz 200^{TM} must go through the spacewash for the advertisement to be true? (Assume that the wash is instantaneous.)

   (b) The pilot of the WizzFlizz 200^{TM} observes the wash process from the spaceship. At the minimum speed v_0, how accurate does the pilot's clock need to be to see that the simultaneity claim is false? (Assuming that it's true if observed from the spacewash.)

   (c) Suppose the spaceship is approaching with insufficient speed $(4/5)v_0$. The spacewash has its own engine that allows it to accelerate toward the ship. To what speed should it accelerate to perform the wash as promised in the advertisement?

4. Consider a simple harmonic oscillator with the Hamiltonian:

$$\hat{H} = \frac{1}{2}m\dot{\hat{x}}^2 + \frac{1}{2}m\omega^2\hat{x}^2, \tag{2}$$

where $\hat{x}(t)$ is the hermitian position operator in the Heisenberg picture. The Heisenberg equation of motion,

$$\ddot{\hat{x}}(t) + \omega^2\hat{x}(t) = 0\,, \tag{3}$$

tells us $\hat{x}(t) \propto e^{\pm i\omega t}$, and thus $\hat{x}(t)$ can be written as

$$\hat{x}(t) = \beta e^{-i\omega t}\hat{a} + \beta^* e^{i\omega t}\hat{a}^\dagger\,, \tag{4}$$

where β is a complex number in general, and $\hat{a}$ and $\hat{a}^\dagger$ are the lowering and raising operators, satisfying the commutator $[\hat{a}, \hat{a}^\dagger] = 1$.

(a) What value should β take in order that $[\hat{x}(t), m\dot{\hat{x}}(t)] = i\hbar$?

(b) Let us denote the ground state by $|0\rangle$ i.e. $\hat{a}|0\rangle = 0$. Find $\langle 0|\hat{x}(t)^2|0\rangle$.

(c) Find $\langle 0|\hat{x}(t_1)\hat{x}(t_2)|0\rangle$, $\langle 0|\,[\hat{x}(t_1), \hat{x}(t_2)]\,|0\rangle$, and $\partial_{t_2}\langle 0|\,[\hat{x}(t_1), \hat{x}(t_2)]\,|0\rangle$. What value does the last quantity take in the limit $t_1 \to t_2$?

(d) Find $\langle 0|\hat{x}(t)^3|0\rangle$, $\langle 0|\hat{x}(t)^4|0\rangle$, and $\langle 0|\hat{x}(t)^6|0\rangle$.

(e) Can you guess what $\langle 0|\hat{x}(t)^n|0\rangle$ is, for a general positive integer n ?

5. Consider a hydrogen atom in its ground state. The hyperfine interaction between the magnetic moment of the proton and the magnetic moment of the electron is described by the Hamiltonian

$$H_{hf} = A\, \boldsymbol{S}_1 \cdot \boldsymbol{S}_2, \tag{5}$$

where $\boldsymbol{S}_1$ is the spin of the electron, $\boldsymbol{S}_2$ is the spin of the proton, and A is a constant. The ground state is split by the hyperfine coupling. Obtain the energies of the split levels.

Problem:
A hoop of radius R is floating in space. In an inertial lab frame, the hoop is observed to rotate about its center with angular velocity $\vec{\Omega}$ perpendicular to the plane of the hoop. Rabbit Carlo of mass m is comfortably sitting on the inner side of the hoop. His sitting is helped by the effective "gravity" due to centrifugal acceleration (like in the rotating spaceship in Stanley Kubrick's "2001: A Space Odyssey"). In classical Newtonian mechanics, Carlo would find that his weight is mv^2/R where $v = \Omega R \ll c$. Obtain the relativistic generalization of this formula that is valid for any v.

Solution:
In the lab frame, Carlo's acceleration is $\vec{a} = -(v^2/R^2)\vec{R}$, where $\vec{R}$ is Carlo's radius-vector. His weight $-m\vec{a}'$ is measured in his rest frame, which is moving with the instantaneous velocity $\vec{v} = \vec{\Omega} \times \vec{R}$. The problem is reduced to finding Carlo's acceleration in the moving frame, $\vec{a}'$, using the known $\vec{v}$ and $\vec{a}$. Note that $\vec{a} \perp \vec{v}$.

One possible strategy is to transform a velocity $\vec{w}$ from the lab frame to Carlo's frame, and differentiate the obtained relation with respect to time to find the transformation of acceleration.

A better way is to write 4-acceleration $A^\mu = du^\mu/d\tau$ where $\mu = 0, 1, 2, 3$, $u^\mu = (c\gamma, \gamma\vec{v})$ is 4-velocity, $d\tau = dt/\gamma$ is the proper time. Using $\gamma = \textit{const}$ along Carlo's worldline, one immediately finds

$$A^\mu = \gamma \frac{du^\mu}{dt} = (0, \gamma^2 \vec{a}) = (0, 0, \gamma^2 a, 0),$$

where $\vec{a} = d\vec{v}/dt$; we chose the x-axis along the instantaneous $\vec{v}$ and the y-axis along $\vec{a}$. A^μ is a 4-vector with a single non-zero component A^y, and the Lorentz boost along the x-axis (to Carlo's frame) does not change it,

$$A^{\mu'} = A^\mu.$$

On the other hand, the definition $A^{\mu'} = du^{\mu'}/d\tau$ gives

$$A^{\mu'} = (0, 0, {\gamma'}^2 a', 0) = (0, 0, a', 0),$$

as $\gamma' = 1$ (Carlo is at rest in the new frame). In summary, $a' = A^{y'} = A^y = \gamma^2 a$, so Carlo's weight is

$$ma' = m\gamma^2 a = \frac{mv^2}{(1 - v^2/c^2)R}.$$

Applied QM/Special Relativity (Metzger)
If the motion of an electron (of charge $-e$) around a nucleus (of charge $+e$ and mass $m \gg m_e$) is constrained in the plane by certain boundary conditions, we call such system a "two-dimensional (2D) hydrogen atom". This problem examines some properties of the 2D hydrogen atom in polar coordinates (r, ϕ). The potential for the system is

$$V(r) = -\frac{e^2}{4\pi\epsilon_0 r}$$

1. In analogy with the three dimensional hydrogen atom, let us propose a trial wavefunction of the form

$$\psi_g(r, \phi) = Ae^{-\alpha r}$$

   for the ground state of the 2D hydrogen atom.
   Using the variation principle with this wavefunction, compute the ground state energy of the 2D hydrogen atom in eV. Compare your answer to the ground state energy of the 3D hydrogen atom.

2. The above trial wavefunction is indeed the correct ground state wavefunction of the 2D hydrogen atom. When the 2D hydrogen atom is solved analytically, the ground state corresponds to quantum numbers $n = 1$, $l = 0$, and is non-degenerate (i.e. $\psi_g = \psi_{1,0}$). The first excited state, on the other hand, is three-fold degenerate with wavefunctions given by $n = 2$, $l = 0, \pm 1$

$$\psi_{2,0}(r, \phi) = \frac{1}{\sqrt{2\pi}} R_{2,0}(r)$$

$$\psi_{2,\pm 1}(r, \phi) = \frac{1}{\sqrt{2\pi}} R_{2,1}(r) e^{\pm i\phi},$$

   where $R_{2,0}(r)$ and $R_{2,1}(r)$ are real radial functions.

   In analogy with the 3D hydrogen atom, the radial part of the energy eigenfunction for the $n = 2, l = 0$ state can be proposed to be of the form

$$R_{2,0}(r, \phi) = B(1 - \beta r)e^{-\gamma r}$$

   Express β in terms of α and γ.
   Note: you are not being asked to evaluated B or γ, or use the calculated value of α. You are just being asked to obtain a simple relationship between α, β, and γ

Solution:

1. First normalize $\psi_g(r)$,

$$A^2 \int |\psi_g(\vec{r})|^2 d^2\vec{r} = 1 \Rightarrow A^2 \cdot 2\pi \int_0^\infty re^{-2\alpha r} dr = 1 \Rightarrow A = \alpha\sqrt{\frac{2}{\pi}}$$

The Hamiltonian for the system is

$$H = -\frac{\hbar^2}{2\mu}\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial}{\partial \phi^2}\right) - \frac{e^2}{4\pi\epsilon_0 r}$$

such that

$$\langle KE\rangle = -\frac{\hbar^2 A^2}{2\mu} \cdot 2\pi \left[\int_0^\infty e^{-\alpha r}(\alpha^2)e^{-\alpha r} r dr + \int_0^\infty e^{-\alpha r}(-\alpha)e^{-\alpha r} dr\right]$$

$$= \frac{\hbar^2 A^2 \pi}{4\mu} = \frac{\hbar^2\alpha^2}{2\mu}$$

$$\langle V\rangle = \frac{-e^2}{4\pi\epsilon_0} A^2 \cdot 2\pi \int_0^\infty e^{-2\alpha r} dr = \frac{-e^2}{8\pi\epsilon_0\alpha} 2\pi \frac{2\alpha^2}{\pi} = -\frac{e^2\alpha}{2\pi\epsilon_0}$$

Thus,

$$\langle H\rangle = \frac{\hbar^2\alpha^2}{2\mu} - \frac{e^2\alpha}{2\pi\epsilon_0}$$

$$\frac{d\langle H\rangle}{d\alpha} = 0 \Rightarrow \frac{\hbar^2\alpha}{\mu} - \frac{e^2}{2\pi\epsilon_0} \Rightarrow \alpha_{\min} = \frac{\mu e^2}{2\pi\hbar^2\epsilon_0}$$

$$E_{\min} = \langle H\rangle_{\alpha_{\min}} + \langle V\rangle_{\alpha_{\min}} = \frac{\hbar^2}{2\mu}\frac{\mu^2 e^4}{4\pi^2\hbar^4\epsilon_0^2} - \frac{e^2}{2\pi\epsilon_0}\frac{\mu e^2}{2\pi\hbar^2\epsilon_0} = -\frac{-\mu e^4}{8\pi^2\hbar^2\epsilon_0^2} = -54.4\text{eV},$$

i.e. 4 times larger than the 3D hydrogen atom.

2. Since $\psi_{1,0}(r,\phi)$ and $\psi_{2,0}(r,\phi)$ have the same angular part (no $\phi-$dependence), their orthgonality with respect to each other is established through orthogonality of $R_{1,0}(r)$ and $R_{2,0}(r)$. This implies that

$$\langle\psi_{1,0}|\psi_{2,0}\rangle = 0 \Rightarrow \int_0^\infty e^{-\alpha r}(1-\beta r)e^{-\gamma r} r dr = \int_0^\infty re^{-(\alpha+\gamma)r} dr - \beta\int_0^\infty r^2 e^{-(\alpha+\beta)r} dr = 0$$

$$\frac{1}{(\alpha+\gamma)^2} - \frac{2\beta}{(\alpha+\gamma)^3} = 0 \Rightarrow 1 - \frac{2\beta}{\alpha+\gamma} = 0 \Rightarrow \beta = \frac{\alpha+\gamma}{2}$$

Brooijmans
Section 4-3

1 Relativity Problem

A young entrepreneur decides to operate a spacewash in Earth orbit. His funds are limited so that the total length of the device is only 100 meters. At the Solar Oasis (a reststop in the solar system) he advertises his spacewash as "the only one that simultaneously washes front and back of a WizzFlizz 200^{TM}". (According to the catalog, the WizzFlizz 200^{TM} is 200 meters long - quite a claim for such a fast ship). a) What is the minimum speed (in units of c) at which such a ship must go through the spacewash for this advertisement to be true if the ship's pilot is to observe the process from a coffeelounge on the spacewash? b) Since the pilot doesn't trust humans, he has left his assistant on the ship. At this minimum speed, how accurate does the assistant's clock need to be to observe that the simultaneity claim is false? (Assuming that it's true from the pilot's point of view.) c) While sipping his coffee, the pilot notices that when the front of his ship reaches the front of the spacewash, the back is still sticking out by 10 meters. Assuming that the catalog is truthful, which may be a stretch, and given that the entrepreneur had the spacewash built on Earth, at what speed is it orbiting the Earth? Be advised that the ship is sticking to the minimum speed suggested to go through the spacewash. (Assume that the spacewash manufacturer "cheated" the entrepreneur by having it measured before its launch, which is anything but a stretch).

1.1 Solution

a) We need the ship to be contracted to 100 m. So

$$\gamma = L_0/L = 2 \Rightarrow 1 - \beta^2 = 1/4 \Rightarrow \beta = \sqrt{3}/2. \tag{1}$$

b) If the events are simultaneous in the pilot's frame, there $\Delta t = 0$. Using the Lorentz transformations yields

$$\Delta t' = \frac{v}{c^2}\gamma\Delta x = \frac{\sqrt{3}}{2c}.2.100m = \frac{\sqrt{3}}{3}10^{-6}s. \tag{2}$$

The assistant's clock must be able to measure time intervals smaller than that.

c) If the back of the ship sticks out by 10 m and the ship is going at the minimum speed, the spacewash must only be 90 m long. If it was 100 m long on Earth, it must have

$$\gamma = L_0/L = 1.1 \Rightarrow 1 - \beta^2 = 0.81 \Rightarrow \beta = \sqrt{0.19}. \tag{3}$$

2 Physical Phenomena Problem

3 Solution

This is for the Quals to be held in January 2017.

QM problem. Consider a simple harmonic oscillator with the Hamiltonian:

$$\hat{H} = \frac{1}{2}m\dot{\hat{x}}^2 + \frac{1}{2}m\omega^2\hat{x}^2 \tag{1}$$

where $\hat{x}(t)$ is the hermitian position operator in the Heisenberg picture. The Heisenberg equation of motion,

$$\ddot{\hat{x}}(t) + \omega^2\hat{x}(t) = 0\,, \tag{2}$$

tells us $\hat{x}(t) \propto e^{\pm i\omega t}$, and thus $\hat{x}(t)$ can be written as

$$\hat{x}(t) = \beta e^{-i\omega t}\hat{a} + \beta^* e^{i\omega t}\hat{a}^\dagger\,, \tag{3}$$

where β is a complex number in general, and $\hat{a}$ and $\hat{a}^\dagger$ are the lowering and raising operators, satisfying the commutator $[\hat{a}, \hat{a}^\dagger] = 1$.

a. What value should β take in order that $[\hat{x}(t), m\dot{\hat{x}}(t)] = i\hbar$?

b. Let us denote the ground state by $|0\rangle$ i.e. $\hat{a}|0\rangle = 0$. What is $\langle 0|\hat{x}(t)^2|0\rangle$?

c. What is $\langle 0|\hat{x}(t_1)\hat{x}(t_2)|0\rangle$? What is $\langle 0|\,[\hat{x}(t_1), \hat{x}(t_2)]\,|0\rangle$? What is $\partial_{t_2}\langle 0|\,[\hat{x}(t_1), \hat{x}(t_2)]\,|0\rangle$? What value does the last quantity take in the limit $t_1 \to t_2$?

d. What are the values of $\langle 0|\hat{x}(t)^3|0\rangle$, $\langle 0|\hat{x}(t)^4|0\rangle$, $\langle 0|\hat{x}(t)^6|0\rangle$?

e. Can you guess what $\langle 0|\hat{x}(t)^n|0\rangle$ is, for a general positive integer n ?

Solution (Lam).

a.

$$[\hat{x}(t), m\dot{\hat{x}}(t)] = im\omega|\beta|^2[\hat{a}, \hat{a}^\dagger] - im\omega|\beta|^2[\hat{a}^\dagger, \hat{a}]\,. \tag{4}$$

Therefore, we want

$$|\beta| = \sqrt{\frac{\hbar}{2m\omega}} \tag{5}$$

while the phase of β is arbitrary.

b.

$$\langle 0|\hat{x}(t)^2|0\rangle = |\beta|^2\,, \tag{6}$$

c.

$$\langle 0|\hat{x}(t_1)\hat{x}(t_2)|0\rangle = |\beta|^2 e^{-i\omega(t_1-t_2)}\,, \tag{7}$$

and

$$\langle 0|[\hat{x}(t_1),\hat{x}(t_2)]|0\rangle = |\beta|^2\left(e^{-i\omega(t_1-t_2)} - e^{i\omega(t_1-t_2)}\right) = -2i|\beta|^2\sin(\omega(t_1-t_2))\,, \tag{8}$$

and

$$\partial_{t_2}\langle 0|[\hat{x}(t_1),\hat{x}(t_2)]|0\rangle = 2i\omega|\beta|^2\cos(\omega(t_1-t_2))\,. \tag{9}$$

In the limit $t_1 \to t_2$, the last quantity gives $i\hbar/m$, as expected.

d.

$$\langle 0|\hat{x}^3(t)|0\rangle = 0\,, \tag{10}$$

and

$$\langle 0|\hat{x}^4(t)|0\rangle = 3|\beta|^4\,, \tag{11}$$

and

$$\langle 0|\hat{x}^6(t)|0\rangle = 15|\beta|^6\,. \tag{12}$$

e. With some guess work, one could arrive at what is essentially Wick's theorem (or its corollary):

$$\langle 0|\hat{x}^n(t)|0\rangle = \frac{n!}{(n/2)!\,2^{n/2}}|\beta|^n \tag{13}$$

if n is even, and vanishes if n is odd. Alternatively, one could derive this by using the fact that probability distribution for x is Gaussian.

General-Section 4: applied quantum mechanics

Solution

Total angular momentum is:

$$\mathbf{S} = \mathbf{S_1} + \mathbf{S_2}$$

For $S_1 = ½$ and $S_2 = ½$ the two states are:

<u>singlet</u> with S=0; and

<u>triplet</u> with S=1

Write:

$$S^2 = (\mathbf{S_1} + \mathbf{S_2})^2 = S_1^2 + S_2^2 + 2\mathbf{S_2.S_2}$$

$$H_{hf} = A\mathbf{S_1.S_2} = (A/2)\,(S^2 - S_1^2 - S_2^2) \qquad (1)$$

In Equation (1) $\mathbf{S_1}$ is the spin of the electron and $\mathbf{S_2}$ is the spin of the proton.

The needed eigenvalues are:

For S^2	$\hbar^2\, S(S+1)$	(2)
For S_1^2	$\hbar^2\, S_1(S_1+1)$	(3)
For S_2^2	$\hbar^2\, S_2(S_2+1)$	(4)

From Equations (1) to (4) it follows that the hyperfine energies are:

for the <u>singlet</u> $-3\hbar^2A/4$

for the <u>triplet</u> $\hbar^2A/4$